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DIFFERENCE OPERATOR ON DOUBLE SEQUENCES WITH AN ORLICZ FUNCTION

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Abstract. In this article we introduce some vector valued difference double sequence spaces defined by Orlicz function. We study some of their properties like symmetricity, completeness etc. and prove some inclusion results.

Key words. Orlicz function, difference space, completeness, seminorm, regular convergence, symmetric space.

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1. Introduction

Throughout the article ${}_2w(q)$, ${}_2\ell_\infty(q)$, ${}_2c(q)$, ${}_2c_0(q)$, ${}_2c^R(q)$, ${}_2c_0^R(q)$ denote the spaces of *all*, *bounded*, *convergent in Pringsheim's sense*, *null in Pringsheim's sense*, *regularly convergent* and *regularly null* double sequences, defined over a seminormed space (X, q), seminormed by q.

An *Orlicz function M* is a mapping $M:[0, \infty) \to [0, \infty)$ such that it is *continuous*, *non-decreasing* and *convex* with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

Throughout, a double sequence is denoted by $A = \langle a_{nk} \rangle$, a double infinite array of elements $a_{nk} \in X$ for all $n, k \in N$.

The initial works on double sequences are found in Bromwich [2]. Later on it is studied by Hardy [3], Moricz [7], Moricz and Rhoades [8], Tripathy [9], Basarir and Sonalcan [1], Tripathy and Sarma ([10], [11]) and many others. Hardy [3] introduced the notion of regular convergence for double sequences.

Definition. A double sequence space E is said to be *symmetric* if $\langle a_{nk} \rangle \in E$ implies $\langle a_{\pi(n)\pi(k)} \rangle \in E$, where π is a permutation of N.

We introduce the following difference double sequence spaces.

$${}_2\ell_\infty(M,\ \Delta,\ q) = \{<a_{nk}> \ \in\ {}_2w(q) :\ \sup_{n,k} M\left(q\left(\frac{\Delta a_{nk}}{\rho}\right)\right) < \infty,\ \text{for some } \rho > 0\}.$$

$${}_2c(M,\ \Delta, q) = \{<a_{nk}> \ \in\ {}_2w(q) :\ \ M\left(q\left(\frac{\Delta a_{nk} - L}{\rho}\right)\right) \to 0,\ \text{as}\ \ n, k \to \infty,$$

for some $\rho > 0\}$, where $\Delta a_{nk} = a_{nk} - a_{n+1,k} - a_{n,k+1} + a_{n+1,k+1}$, for all $n, k \in N$.

Also $<a_{nk}> \ \in\ {}_2c^R\ (M,\ \Delta,\ q)$ *i.e.* regularly convergent if $<a_{nk}> \in {}_2c(M,\ \Delta, q)$ and the following limits hold:

There exists $L_k \in X$, such that $M\left(q\left(\frac{\Delta a_{nk} - L_k}{\rho}\right)\right) \to 0$, as $n \to \infty$, for some $\rho > 0$ and all $k \in N$.

There exists $J_n \in X$, such that $M\left(q\left(\frac{\Delta a_{nk} - J_n}{\rho}\right)\right) \to 0$, as $k \to \infty$, for some $\rho > 0$ and all $n \in N$.

The definitions of ${}_2c_0(M,\ \Delta,\ q)$ and ${}_2c_0^R\ (M,\ \Delta,\ q)$ follow from the above definition on taking $L = L_k = J_n = \theta$, for all $n, k \in N$.

3. Main Results

Theorem 3.1. *The classes $Z(M,\ \Delta,\ q)$, where $Z = {}_2c, {}_2c_0, {}_2c^R, {}_2c_0^R$ and ${}_2\ell_\infty$ are linear spaces.*

Theorem 3.2. *Let $(X,\ q)$ be a complete seminormed space. Then the spaces $Z(M,\ \Delta,\ q)$, where $Z = {}_2c^R, {}_2c_0^R$ and ${}_2\ell_\infty$ are complete seminormed spaces seminormed by*

$$f(<a_{nk}>) = \inf\left\{\rho > 0 : \sup_n M\left(q\left(\frac{a_{n1}}{\rho}\right)\right) + \sup_k M\left(q\left(\frac{a_{1k}}{\rho}\right)\right) + \sup_{n,k} M\left(q\left(\frac{\Delta a_{nk}}{\rho}\right)\right) \le 1\right\}$$

Proof. Let us consider the space ${}_2\ell_\infty(M, \Delta, q)$. It is simple to prove that ${}_2\ell_\infty(M, \Delta, q)$ is a seminormed space. Let $< a_{nk}^i >$ be a Cauchy sequence in ${}_2\ell_\infty(M, \Delta, q)$. Then for fixed x_0, $r > 0$, we have

$$f(< a_{nk}^i - a_{nk}^j >) < \frac{\varepsilon}{r x_0} \quad \text{for all}\ \ i,\ j \ge m_0,\ (m_0 \in N).$$

Also for $r > 0$, choose $M\left(\frac{r x_0}{2}\right) \ge 1$.

By the definition of the seminorm

$$\sup_n M\left(q\left(\frac{a_{n1}^i - a_{n1}^j}{\rho}\right)\right) + \sup_k M\left(q\left(\frac{a_{1k}^i - a_{1k}^j}{\rho}\right)\right) + \sup_{n,k} M\left(q\left(\frac{\Delta a_{nk}^i - \Delta a_{nk}^j}{\rho}\right)\right) \le 1$$

$$\le M\left(\frac{rx_0}{2}\right) \quad \ldots. \ \ldots. \ \ldots. \qquad (1)$$

$$\Rightarrow \ M\left(q\left(\frac{a_{n1}^i - a_{n1}^j}{f(< a_{nk}^i - a_{nk}^j >)}\right)\right) \le M\left(\frac{rx_0}{2}\right), \ M\left(\left(\frac{a_{1k}^i - a_{1k}^j}{f(< a_{nk}^i - a_{nk}^j >)}\right)\right) \le M\left(\frac{rx_0}{2}\right)$$

$$\text{and} \quad M\left(q\left(\frac{\Delta a_{nk}^i - \Delta a_{nk}^j}{f(< a_{nk}^i - a_{nk}^j >)}\right)\right) \le M\left(\frac{rx_0}{2}\right)$$

$$\Rightarrow \ q\left(a_{n1}^i - a_{n1}^j\right) < \frac{rx_0}{2}.\frac{\varepsilon}{rx_0} = \frac{\varepsilon}{2} \quad \text{for all} \quad i, j \ge m_0.$$

$$q\left(a_{1k}^i - a_{1k}^j\right) < \frac{rx_0}{2}.\frac{\varepsilon}{rx_0} = \frac{\varepsilon}{2} \quad \text{for all} \quad i, j \ge m_0.$$

$$q\left(\Delta a_{nk}^i - \Delta a_{nk}^j\right) < \frac{rx_0}{2}.\frac{\varepsilon}{rx_0} = \frac{\varepsilon}{2} \quad \text{for all} \quad i, j \ge m_0.$$

Thus $< a_{n1}^i >$, $< a_{1k}^i >$ and $< \Delta a_{nk}^i >$ are Cauchy sequences in X. Since X is complete so there exist $a_{n1}, \ a_{1k}, \ y_{nk} \in X$ such that

$$\lim_{i\to\infty} a_{n1}^i = a_{n1}, \ \lim_{i\to\infty} a_{1k}^i = a_{1k} \quad \text{and} \quad \lim_{i\to\infty} \Delta a_{nk}^i = y_{nk}.$$

From this it is clear that $\lim_{i\to\infty} a_{nk}^i \in X$.

Since M is continuous so taking $j \to \infty$ in (1) we get

$$\sup_n M\left(q\left(\frac{a_{n1}^i - a_{n1}}{\rho}\right)\right) + \sup_k M\left(q\left(\frac{a_{1k}^i - a_{1k}}{\rho}\right)\right) + \sup_{n,k} M\left(q\left(\frac{\Delta a_{nk}^i - \Delta a_{nk}}{\rho}\right)\right) \le 1$$

Taking infimum of such ρ's, we get

$$\inf\left\{\rho : \sup_n M\left(q\left(\frac{a_{n1}^i - a_{n1}}{\rho}\right)\right) + \sup_k M\left(q\left(\frac{a_{1k}^i - a_{1k}}{\rho}\right)\right) + \sup_{n,k} M\left(q\left(\frac{\Delta a_{nk}^i - \Delta a_{nk}}{\rho}\right)\right) \le 1\right\} < \varepsilon$$

$$\text{for all} \quad i, j \ge m_0.$$

Hence $< a_{nk}^i - a_{nk} > \in {}_2\ell_\infty(M, \Delta, q)$. Since ${}_2\ell_\infty(M, \Delta, q)$ is linear, so $< a_{nk}^i - a_{nk} >, < a_{nk}^i > \in {}_2\ell_\infty(M, \Delta, q)$ implies $< a_{nk} > \, = \, < a_{nk} - a_{nk}^i + a_{nk}^i > \in {}_2\ell_\infty(M, \Delta, q)$.

Thus ${}_2\ell_\infty(M, \Delta, q)$ is complete. The other cases can be proved using similar technique.

Proposition. 3.4. (i) $Z(M, \Delta, q) \subseteq {}_2\ell_\infty(M, \Delta, q)$ for $Z = {}_2c^R$, ${}_2c_0^R$. *The inclusions are strict.*

(ii) $Z(M, q) \subseteq Y(M, \Delta, q)$ for $Z = {}_2c$, ${}_2c^R$ *and* $Y = {}_2c_0, {}_2c_0^R$ *respectively. The inclusions are strict.*

Proof. (i) The first part is obvious. To show the inclusions are strict, consider the following example.

Example 3.2. Let $X = C$, $M(x) = x$ and $q(x) = |x|$. Let the sequence $<a_{nk}>$ be defined by

$$a_{nk} = n + k, \text{ for } n \text{ odd and all } k \in N.$$

$$= 0, \text{ otherwise.}$$

Then $< a_{nk} > \in {}_2\ell_\infty(M, \Delta, q)$ but $< a_{nk} > \notin Z(M, \Delta, q)$ for $Z = {}_2c^R$, ${}_2c_0^R$.

(ii) We prove ${}_2c(M, q) \subseteq {}_2c_0(M, \Delta, q)$. Let $< a_{nk} > \in {}_2c(M, q)$. Then for some $\rho > 0$,

$$M\left(q\left(\frac{a_{nk} - L}{\rho}\right)\right) < \varepsilon \text{ for all } n \geq n_0,\ k \geq k_0,\ (n_0, k_0 \in N).$$

Let $r = 4\rho$. Now

$$M\left(q\left(\frac{\Delta a_{nk}}{r}\right)\right) = M\left(q\left(\frac{a_{nk} - a_{n+1,k} - a_{n,k+1} + a_{n+1,k+1}}{r}\right)\right)$$

$$= M\left(q\left(\frac{(a_{nk} - L) + (-a_{n+1,k} + L) + (-a_{n,k+1} + L) + (a_{n+1,k+1} - L)}{r}\right)\right)$$

$$\leq M\left(q\left(\frac{a_{nk} - L}{4\rho}\right) + q\left(\frac{a_{n+1,k} - L}{4\rho}\right) + q\left(\frac{a_{n,k+1} - L}{4\rho}\right) + q\left(\frac{a_{n+1,k+1} - L}{4\rho}\right)\right)$$

$$\leq \frac{1}{4}M\left(q\left(\frac{a_{nk} - L}{\rho}\right)\right) + \frac{1}{4}M\left(q\left(\frac{a_{n,k+1} - L}{\rho}\right)\right) + \frac{1}{4}M\left(q\left(\frac{a_{n,k+1} - L}{\rho}\right)\right) + \frac{1}{4}M\left(q\left(\frac{a_{n+1,k+1} - L}{\rho}\right)\right)$$

$< \varepsilon$ for all $n \geq n_0,\ k \geq k_0,\ (n_0, k_0 \in N)$ and for some $r > 0$.

Hence $< a_{nk} > \in {}_2c_0(M, \Delta, q)$. Thus ${}_2c(M, q) \subseteq {}_2c_0(M, \Delta, q)$. Similarly it can be proved that ${}_2c^R(M, q) \subseteq {}_2c_0^R(M, \Delta, q)$.

To show the strict inclusions, consider the following example.

Example 3.3. Let $X = C$, $M(x) = x$ and $q(x) = |x|$. Let the sequence $<a_{nk}>$ be defined by

$$a_{nk} = n + k, \text{ for all } n, k \in N.$$

Clearly $<a_{nk}>\in {}_2c(M,\Delta,q)$ but $<a_{nk}>\notin {}_2c(M,q)$.

Proposition 3.5. *Let the Orlicz functions* M, M_1, M_2 *satisfy the* Δ_2*-conditions, then*

(i) $Z(M_2,\Delta,q)\subseteq Z(M_1,\Delta,q)$ *for* $Z={}_2c,\ {}_2c_0,\ {}_2c^R,\ {}_2c_0^R$ *if* $M_1(x)\le M_2(x)$ *for all* $x\in[0,\infty)$.

(ii) $Z(M_1,\Delta,q)\subseteq Z(M\circ M_1,\Delta,q)$ *for* $Z={}_2c,\ {}_2c_0,\ {}_2c^R,\ {}_2c_0^R$.

(iii) $Z(M_1,\Delta,q)\cap Z(M_2,\Delta,q)\subseteq Z(M_1+M_2,\Delta,q)$ *for* $Z={}_2c,\ {}_2c_0,\ {}_2c^R,\ {}_2c_0^R$.

Proof. (i) The proof is obvious.

(ii) Consider $Z={}_2c$. Let $<a_{nk}>\in {}_2c(M_1,\Delta,q)$. Then for some $\rho>0$,

$$M_1\left(q\left(\frac{\Delta a_{nk}-L}{\rho}\right)\right)\to 0 \text{ as } n\to\infty,\ k\to\infty.$$

Let $b_{nk}=M_1\left(q\left(\frac{\Delta a_{nk}-L}{\rho}\right)\right)$. Since $b_{nk}\to 0$, there exists $n_0,k_0\in N$ such that $b_{nk}<1$ for all $n\ge n_0,\ k\ge k_0$.

Now by the remark, $M(b_{nk})\le b_{nk}M(1)$, for all $n\ge n_0,\ k\ge k_0$.

Thus $M\left(M_1\left(q\left(\frac{\Delta a_{nk}-L}{\rho}\right)\right)\right)\to 0$ as $n\to\infty,\ k\to\infty$

and this implies that

$$(M\circ M_1)\left(q\left(\frac{\Delta a_{nk}-L}{\rho}\right)\right)\to 0 \text{ as } n\to\infty,\ k\to\infty.$$

Hence $<a_{nk}>\in (M\circ M_1,\Delta,q)$. Similarly the result can be proved for the other spaces also.

(iii) Consider the case for $Z={}_2c$. Let $<a_{nk}>\in {}_2c(M_1,\Delta,q)\cap {}_2c(M_2,\Delta,q)$. Then for some $\rho_1,\rho_2>0$,

$$M_1\left(q\left(\frac{\Delta a_{nk}-L}{\rho_1}\right)\right)<\frac{\varepsilon}{2} \text{ for all } n\ge n_0,\ k\ge k_0,\ (n_0,k_0\in N).$$

$$M_2\left(q\left(\frac{\Delta a_{nk}-L}{\rho_2}\right)\right)<\frac{\varepsilon}{2} \text{ for all } n\ge n_0',\ k\ge k_0',\ (n_0',k_0'\in N).$$

Let $\rho = \max \{\rho_1, \rho_2\}$, $n_0^{//} = \max\{n_0, n_0^{/}\}$, $k_0^{//} = \max\{k_0, k_0^{/}\}$.

Now for $n \geq n_0^{//}$, $k \geq k_0^{//}$ and for some $\rho > 0$,

$$(M_1 + M_2)\left(q\left(\frac{\Delta a_{nk} - L}{\rho}\right)\right) \leq M_1\left(q\left(\frac{\Delta a_{nk} - L}{\rho_1}\right)\right) + M_2\left(q\left(\frac{\Delta a_{nk} - L}{\rho_2}\right)\right)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Thus $< a_{nk} > \in {}_2c(M_1 + M_2, \Delta, q)$. Hence the proof is complete. Similarly it can be proved for the other spaces also.

References

[1] Basarir, M. and Sonalcan, O.: On some double sequence spaces; *J. Indian Acad. Math.* 21(2), (1999); 193-200.

[2] Bromwich T.J.IA: *An Introduction to the Theory of Infinite Series*; Macmillan and Co. Ltd. New York (1965).

[3] Hardy G. H.: On the convergence of certain multiple series; *Proc. Camb. Phil. Soc.*; 9 (1917), 86-95.

[4] Kamthan, P.K. and Gupta, M.: *Sequence Spaces and Series*: Marcel Dekker, 1980.

[5] Kizmaz, H: On certain sequence spaces; *Canad. Math. Bull.*, 24 (1981), 169–176.

[6] Lindenstrauss, J. and Tzafriri, L.: On Orlicz sequence spaces: *Israel J. Math.* 10 (1971), 379-390.

[7] Moricz, F: Extension of the spaces c and c_0 from single to double sequences; *Acta. Math. Hungerica.*; 57(1-2), (1991), 129-136.

[8] Moricz, F. and Rhoades B.E.: Almost convergence of double sequences and strong regularity of summability Matrices; *Math. Proc. Camb. Phil. Soc.*; 104 (1988), 283-294.

[9] Tripathy B.C.: A class of difference sequences related to p-normed space ℓ^p; *Demonstratio Math.* 36 (4) (2003). 867-872.

[10] Tripathy B.C. and Sarma B.: Vector valued paranormed statistically convergent double sequence spaces; *Mathematica Slovaca*; 57(2) (2007), 179-188.

[11] Tripathy B.C. and Sarma B.: Statistically convergent double sequence spaces defined by Orlicz function; *Soochow Journal of Mathematics*; 32(2), (2005), 211-221.