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STRONGLY λ-SUMMABLE SEQUENCES OF İNTERVAL NUMBERS

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Absract: In this paper we introduce and study the concept of strongly λ-summable sequences of interval numbers and discuss some relationships strongly λ-summable sequences and λ-statistical sequences of interval numbers.

Keywords: Interval number, statistical convergence, λ −convergence.
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1. Introduction

Most of mathematical structures have been constructed with real or complex numbers. In recent years, these mathematical structures were replaced by fuzzy numbers or interval numbers and these structures have been very popular since the year 1965. Also, since 1986 these mathematical structures were replaced intuitionicstic fuzzy numbers or two dimensional interval numbers. Interval arithmetic serves as a methodology to analyze and control numerical errors in computers and is widely used in real-valued numerical calculations, particularly those related to rounded and truncated errors appear in machine computations. The topic of interval analysis has been studied for a long time. For a detailed discussion we may suggest refer to some books, for example Moore [18]. The main issue is to regard to closed intervals as a kind of " points". Hereafter we will called them " interval numbers".

Interval arithmetic was first suggested by Dwyer [11] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [18] in 1959 and Moore and Yang [13] in 1962. Furthermore, Moore and others [18], [12] and [14] have developed applications to differential equations.

2. Preliminaries

Chiao in [6] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Sengönül and Eryilmaz in [15] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete metric space. Recently, Esi in [1, 2, 3, 4, 5, 6], Esi and Braha [7], Esi and Esi [8], Esi

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and Hazarika [9] defined and studied different properties of interval numbers.

We denote the set of all real valued closed intervals by IR. Any elements of IR is called interval number and denoted by $\overline{x} = [x_l, x_r]$. Let x_l and x_r be first and last points of x interval number, respectively. For $\overline{x}_1, \overline{x}_2 \in$I$R$, we have $\overline{x}_1 = \overline{x}_2 \Leftrightarrow\ x_{1_l} = x_{2_l},\ x_{1_r} = x_{2_r}$. $\overline{x}_1 + \overline{x}_2 = \{x \in R:\ x_{1_l} + x_{2_l}\ \le\ x \le x_{1_r} + x_{2_r}\}$, and if $\alpha \ge 0$, then $\alpha\overline{x} = \{x \in R:\ \alpha x_{1_l} \le x \le \alpha x_{1_r}\}$ and if $\alpha < 0$, then $\alpha\overline{x} = \{x \in R: \alpha x_{1_r} \le x \le \alpha x_{1_l}\}$,

$\overline{x}_1.\overline{x}_2 = \{x \in R: \min\{x_{1_l}.\ x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}\} \le x \le \max\{x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}\}$.

The set of all interval numbers IR is a complete metric space defined by

$d(\overline{x}_1, \overline{x}_2) = \max\{|x_{1_l} - x_{2_l}|, |x_{1_r} - x_{2_r}|\}$[11].

In the special case $\overline{x}_1 = [a, a]$ and $\overline{x}_2 = [b, b]$, we obtain usual metric of R. Let us define transformation $f: N \to R$ by $k \to f(k) = \overline{x}_k$, $\overline{x} = (\overline{x}_k)$. Then $\overline{x} = (\overline{x}_k)$ is called sequence of interval numbers. The $\overline{x}_k$ is called k^{th} term of sequence $\overline{x} = (\overline{x}_k)$. Let w^idenotes the set of all interval numbers with real terms and the algebraic properties of w^i can be found in [14].

Now we recall the definition of convergence of interval numbers:

Definition 1: *[10] A sequence $\overline{x} = (\overline{x}_k)$ of interval numbers is said to be convergent to the interval number $\overline{x}_0$ if for each $\varepsilon > 0$ there exists a positive integer k_o such that $d(\overline{x}_k, \overline{x}_o) < \varepsilon$ for all $k \ge k_o$ and we denote it by $lim_k\overline{x}_k = \overline{x}_0$.*

Thus, $\lim_k\overline{x}_k = \overline{x}_o \Leftrightarrow \lim_k x_{k_l} = x_{o_l}$ and $\lim_k x_{k_r} = x_{o_r}$.

Definition 2: *An interval sequence $\overline{x} = (\overline{x}_n)$ is said to be bounded if $sup_n\overline{d}(\overline{x}_n, \overline{0}) < \infty$, equivalently, if there exist $\mu \in \mathbb{R}$ such that $|\overline{x}_n| \le \mu$ for all $n \in \mathbb{N}$.*

We now quote the following definitions which will be needed in the article.

Chiao [10] introduced sequence of interval number sequences and studied some properties. Recently, Dutta and Tripathy [19] introduced and studied the class of p-absolutely sequence space of interval numbers as follows.

$$l^i(p) = \left\{\overline{x} = (\overline{x}_n) \in w^i: \sum_{n=1}^{\infty} \left\{\overline{d}\left(\overline{x}_n, \overline{0}\right)\right\}^{p_n} < \infty\right\},$$

where $p = (p_n)$ is bounded sequence of positive numbers so that $0 < p_n \le H = \sup p_n < \infty$ and the metric $\overline{d}$ on $l^i(p)$ is defined by

$$\overline{d}\left(\overline{x}_n, \overline{y}_n\right) = \left\{\sum_n \left\{\max\{|x_{n_l} - y_{n_l}|, |x_{n_r} - y_{n_r}|\}\right\}^{p_n}\right\}^{\frac{1}{M}}$$

where M=$\max(1, \sup p_n)$.Before giving some relations we will give following definition.

Definition 3: *Let $\lambda = (\lambda_n)$ be non-decreasing sequence of positive numbers tending to infinity such that*

$\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1.$

The generalized de la Vallee-Pousin mean is defined by

$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k$

where $I_n = [n - \lambda_n + 1, n]$. The collection of such sequence $\lambda = (\lambda_n)$ will be denoted by ∇.
Let $K \subseteq \mathbb{N}$ be a set of positive integers. Then

$\delta_\lambda(K) = \lim_n \frac{1}{\lambda_n} |\{n - \lambda_n + 1 \leq k \leq n : k \in K\}|$

is said to be λ −density of K. In case $\lambda_n = n$, the λ −density reduces to natural density.
Esi [1] introduced the λ-statistical convergence for interval number sequences as follows: An interval number sequence $\overline{x} = (\overline{x}_n)$ is said to be $\lambda -$ *statistically convergent* or S_λ^i −convergent to the interval number $\overline{x}_o$ if for every $\varepsilon > 0$

$\lim_n \frac{1}{\lambda_n} \left|\{k \in I_n : \overline{d}(x_k, \overline{x}_o) \geq \varepsilon\}\right|.$

Definition 4 *Let $\lambda = (\lambda_n) \in \nabla$ and $\overline{x} = (\overline{x}_n)$ be a sequence of interval numbers. The sequence $\overline{x} = (\overline{x}_n)$ is said to be strongly λ −summable if there is a interval number $\overline{x}_o$ such that*

$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \{\overline{d}(\overline{x}_k, \overline{x}_o)\}^{p_k} = 0.$

If $\lambda_n = n$, then strongly $\lambda -$summable reduces to strongly Cesaro summable defined as follows:

$\lim_n \frac{1}{n} \sum_{k=1}^{n} \{\overline{d}(\overline{x}_k, \overline{x}_o)\}^{p_k} = 0.$

Theorem 1: *The class of strongly $\lambda -$ summable sequences is linear space with the co-ordinate wise addition and scalar multiplication.*

Proof. Let $\overline{x} = (\overline{x}_n), \overline{y} = (\overline{y}_n)$ be two sequences of interval numbers such that

$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \{\overline{d}(\overline{x}_k, \overline{x}_o)\}^{p_k} = 0$ and $\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \{\overline{d}(\overline{y}_k, \overline{y}_o)\}^{p_k} = 0.$

$\frac{1}{\lambda_n} \sum_{k \in I_n} \{\overline{d}(\overline{x}_k + \overline{y}_k, \overline{x}_o + \overline{y}_o)\}^{p_k} \leq \frac{C}{\lambda_n} \sum_{k \in I_n} \{\overline{d}(\overline{x}_k, \overline{x}_o)\}^{p_k} + \frac{C}{\lambda_n} \sum_{k \in I_n} \{\overline{d}(\overline{y}_k, \overline{y}_o)\}^{p_k}$

where C=max$(1, 2^{H-1})$. Let α be a scalar, then

$\frac{1}{\lambda_n} \sum_{k \in I_n} \{\overline{d}(\alpha\overline{x}_k, \alpha\overline{x}_o)\}^{p_k} \leq |\alpha|^H \frac{1}{\lambda_n} \sum_{k \in I_n} \{\overline{d}(\overline{x}_k, \overline{x}_o)\}^{p_k}.$

This completes the proof.

Definition 5: *Now we exhibit some connections between strongly $\lambda-$summable and $\lambda-$statistically convergence of interval number sequences as follows.*

Theorem 2: *If a sequence $\overline{x}=(\overline{x}_n)$ of interval numbers is strongly $\lambda-$summable to interval number $\overline{x}_o$, then it is $\lambda-$statistically convergent to $\overline{x}_o$.*

Proof. Let $\varepsilon>0$. Since

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left\{\overline{d}(\overline{x}_k,\overline{x}_o)\right\}^{p_k}\geq\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ \overline{d}(x_k,\overline{x}_o)\geq\varepsilon}}\left\{\overline{d}(\overline{x}_k,\overline{x}_o)\right\}^{p_k}\geq\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ \overline{d}(x_k,\overline{x}_o)\geq\varepsilon}}\min(\varepsilon^h,\varepsilon^H)$$

$$\geq\frac{1}{\lambda_n}\left|\left\{k\in I_n:\overline{d}(x_k,\overline{x}_o)\geq\varepsilon\right\}\right|.\min(\varepsilon^h,\varepsilon^H),$$

where $0<h=\min p_n<p_n\leq H=\sup p_n<\infty$. Hence $\overline{x}=(\overline{x}_n)$ is $\lambda-$ *statistically convergent to* $\overline{x}_o$.

We have

Theorem 3: *If a bounded sequence $\overline{x}=(\overline{x}_n)$ of interval numbers is $\lambda-$statistically convergent to $\overline{x}_o$, then it is strongly $\lambda-$summable to interval number $\overline{x}_o$ and hence it is strongly Cesaro summable to $\overline{x}_o$.*

Proof . Let $\varepsilon>0$. Suppose that $\overline{x}=(\overline{x}_n)$ is bounded and $\lambda-$*statistically convergent to*$\overline{x}_o$. Since $\overline{x}=(\overline{x}_n)$ is bounded we write $\overline{d}(\overline{x}_n,\overline{x}_o)\leq T$ for all $n\in\mathbb{N}$. We have

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left\{\overline{d}(\overline{x}_k,\overline{x}_o)\right\}^{p_k}=\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ \overline{d}(x_k,\overline{x}_o)\geq\varepsilon}}\left\{\overline{d}(\overline{x}_k,\overline{x}_o)\right\}^{p_k}+\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ \overline{d}(x_k,\overline{x}_o)<\varepsilon}}\left\{\overline{d}(\overline{x}_k,\overline{x}_o)\right\}^{p_k}$$

$$\leq\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ \overline{d}(x_k,\overline{x}_o)\geq\varepsilon}}\max(T^h,T^H)+\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ \overline{d}(x_k,\overline{x}_o)<\varepsilon}}\varepsilon^{p_k}\leq\max(T^h,T^H)\frac{1}{\lambda_n}\left|\left\{k\in I_n:\overline{d}(x_k,\overline{x}_o)\geq\varepsilon\right\}\right|$$

$$+\max(\varepsilon^h,\varepsilon^H).$$

Hence $\overline{x}=(\overline{x}_n)$ is strongly $\lambda-$summable to interval number $\overline{x}_o$. Further we have

$$\frac{1}{n}\sum_{k=1}^{n}\left\{\overline{d}(\overline{x}_k,\overline{x}_o)\right\}^{p_k}=\frac{1}{n}\sum_{k=1}^{n-\lambda_n}\left\{\overline{d}(\overline{x}_k,\overline{x}_o)\right\}^{p_k}+\frac{1}{n}\sum_{k\in I_n}\left\{\overline{d}(\overline{x}_k,\overline{x}_o)\right\}^{p_k}$$

$$\leq\frac{1}{\lambda_n}\sum_{k=1}^{n-\lambda_n}\left\{\overline{d}(\overline{x}_k,\overline{x}_o)\right\}^{p_k}+\frac{1}{\lambda_n}\sum_{k\in I_n}\left\{\overline{d}(\overline{x}_k,\overline{x}_o)\right\}^{p_k}\leq\frac{2}{\lambda_n}\sum_{k\in I_n}\left\{\overline{d}(\overline{x}_k,\overline{x}_o)\right\}^{p_k}.$$

Then $\overline{x}=(\overline{x}_n)$ is strongly Cesaro summable to $\overline{x}_o$.

Theorem 4: *If a interval numbers sequence $\overline{x}=(\overline{x}_n)$ is statistically convergent to a interval number $\overline{x}_o$ and lim inf$_n$ $\frac{\lambda_n}{n}>0$ then it is $\lambda-$statistically convergent to $\overline{x}_o$.*

Proof. Let $\varepsilon>0$. We have

$$\left\{k\leq n:\overline{d}(x_k,\overline{x}_o)\geq\varepsilon\right\}\supset\left\{k\in I_n:\overline{d}(x_k,\overline{x}_o)\geq\varepsilon\right\}.$$

Therefore,

$\frac{1}{n}\left|\left\{k \in I_n: \overline{d}(x_k, \overline{x}_o) \geq \varepsilon\right\}\right| \geq \frac{1}{n}\left|\left\{k \leq n: \overline{d}(x_k, \overline{x}_o) \geq \varepsilon\right\}\right| \geq \frac{\lambda_n}{n} \cdot \frac{1}{\lambda_n}\left|\left\{k \in I_n: \overline{d}(x_k, \overline{x}_o) \geq \varepsilon\right\}\right|.$

Taking limit as $n \to \infty$ and using $lim\ inf_n \frac{\lambda_n}{n} > 0,$ we get the result.

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