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ON CERTAIN DERIVATIVES OF THE I-FUNCTION OF TWO VARIABLES

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Abstract: In this paper we establish some differentiation formulas for the I-function of two variables recently defined by Shantha Kumari, et al. in [17]. A number of special cases of our results are also discussed.
Keywords: I-function, Mellin Barnes Contour integral, H-function.

1. Introduction

The well known H-function of one variable, introduced by [5] and studied by Braaksma [3] contains as particular cases most of the commonly used special functions of applied mathematics. But it does not contain some of the important functions such as the Riemann zeta functions, polylogarithms etc. By demonstrating several examples of functions which are not included in the Fox's H-function, in 1997, Rathie [16] introduced a new function in the literature namely the "I-function" which is useful in Mathematics, Physics and other branches of applied mathematics. The newly defined function contains the polylogarithms, the exact partition of Gaussian free energy model from statistical mechanics, Feynmann integrals and functions useful in testing hypothesis from statistics as special cases. Recently, the I-function introduced by Rathie [16] has found useful applications in wireless communications [2]. Motivated by this, very recently, Shantha Kumari et al. [17], introduced and studied, I-function of two variables which gives a natural generalization of the H-function of two variables introduced by Mittal and Gupta [13].

The I-function of two variables defined and studied by Shantha Kumari et al. [17] is represented by means of the double Mellin Barnes contour integral in the following manner.

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$I[z_1, z_2]$

$= I^{\,0, n_1 : m_2, n_2 ; m_3, n_3}_{\,p_1, q_1 : p_2, q_2 ; p_3, q_3} \left[\begin{matrix} z_1 \\ z_2 \end{matrix} \middle| \begin{matrix} (a_j; \alpha_j, A_j; \xi_j)_{1,p_1} : (c_j, C_j; U_j)_{1,p_2} ; (e_j, E_j; P_j)_{1,p_3} \\ (b_j; \beta_j, B_j; \eta_j)_{1,q_1} : (d_j, D_j; V_j)_{1,q_2} ; (f_j, F_j; Q_j)_{1,q_3} \end{matrix} \right]$

$$= \frac{1}{(2\pi i)^2} \int_{L_s} \int_{L_t} \phi(s,t)\, \theta_1(s) \theta_2(t)\, ds\, dt \tag{1.1}$$

where $\phi(s,t)$, $\theta_1(s)$ and $\theta_2(t)$ are given by

$$\phi(s,t) = \frac{\prod_{j=1}^{n_1} \Gamma^{\xi_j}(1 - a_j + \alpha_j s + A_j t)}{\prod_{j=n_1+1}^{p_1} \Gamma^{\xi_j}(a_j - \alpha_j s - A_j t) \prod_{j=1}^{q_1} \Gamma^{\eta_j}(1 - b_j + \beta_j s + B_j t)} \tag{1.2}$$

$$\theta_1(s) = \frac{\prod_{j=1}^{n_2} \Gamma^{U_j}(1 - c_j + C_j s) \quad \prod_{j=1}^{m_2} \Gamma^{V_j}(d_j - D_j s)}{\prod_{j=n_2+1}^{p_2} \Gamma^{U_j}(c_j - C_j s) \quad \prod_{j=m_2+1}^{q_2} \Gamma^{V_j}(1 - d_j + D_j s)} \tag{1.3}$$

$$\theta_2(t) = \frac{\prod_{j=1}^{n_3} \Gamma^{P_j}(1 - e_j + E_j t) \quad \prod_{j=1}^{m_3} \Gamma^{Q_j}(f_j - F_j t)}{\prod_{j=n_3+1}^{p_3} \Gamma^{P_j}(e_j - E_j t) \quad \prod_{j=m_3+1}^{q_3} \Gamma^{Q_j}(1 - f_j + F_j t)} \tag{1.4}$$

Also:

- $z_1 \neq 0$, $z_2 \neq 0$;
- $i = \sqrt{-1}$;
- an empty product is interpreted as unity ;
- the parameters n_j, p_j, q_j $(j = 1,2,3)$, $m_j (j = 2,3)$ are nonnegative integers such that $0 \leq n_j \leq p_j$ $(j = 1,2,3)$, $q_1 \geq 0$, $0 \leq m_j \leq q_j (j = 2,3)$ (not all zero simulataneously);
- α_j, A_j $(j = 1, \ldots, p_1)$, β_j, B_j $(j = 1, \ldots, q_1)$, C_j $(j = 1, \ldots, p_2)$, D_j $(j = 1, \ldots, q_2)$, E_j $(j = 1, \ldots, p_3)$, F_j $(j = 1, \ldots, q_3)$ are assumed to be positive quantities for standardisation purpose;
- $a_j (j = 1, \ldots, p_1)$, $b_j (j = 1, \ldots, q_1)$, c_j $(j = 1, \ldots, p_2)$, d_j $(j = 1, \ldots, q_2)$, e_j $(j = 1, \ldots, p_3)$ and f_j $(j = 1, \ldots, q_3)$ are complex numbers;
- The exponents ξ_j $(j = 1, \ldots, p)$, η_j $(j = 1, \ldots, q)$, U_j $(j = 1, \ldots, p_2)$, V_j $(j = 1, \ldots, q_2)$, $P_j (j = 1, \ldots, p_3)$, Q_j $(j = 1, \ldots, q_3)$ of various gamma functions involved in (1.2), (1.3) and (1.4) may take non-integer values.
- L_s and L_t are suitable contours of Mellin - Barnes type. Moreover, the contour L_s is in the complex s-plane and runs from $\sigma_1 - i\infty$ to $\sigma_1 + i\infty$ (σ_1 real) so that all the singularities of $\Gamma^{V_j}(d_j - D_j s) (j = 1, \ldots, m_2)$ lie to the right of L_s and all the singularities of $\Gamma^{U_j}(1 - c_j + C_j s)$ $(j = 1, \ldots, n_2)$, $\Gamma^{\xi_j}(1 - a_j + \alpha_j s + A_j t)$ $(j =$

$1, \ldots, n_1)$ lie to the left of L_s . The other contour L_t follows similar conditions in the complex t-plane.

The function defined by (1.1) is an analytic function of z_1 and z_2 if

$$(i)\ R = \left[\sum_{j=1}^{p_1} \xi_j \alpha_j + \sum_{j=1}^{p_2} U_j C_j - \sum_{j=1}^{q_1} \eta_j \beta_j - \sum_{j=1}^{q_2} V_j D_j\right] < 0 \tag{1.5}$$

$$(ii)\ S = \left[\sum_{j=1}^{p_1} \xi_j A_j + \sum_{j=1}^{p_3} P_j E_j - \sum_{j=1}^{q_1} \eta_j B_j - \sum_{j=1}^{q_3} Q_j F_j\right] < 0 \tag{1.6}$$

Further, the integral (1.1) is convergent if

$$(iii)\ \Delta_1 = \left[\sum_{j=1}^{n_1} \xi_j \alpha_j - \sum_{j=n_1+1}^{p_1} \xi_j \alpha_j - \sum_{j=1}^{q_1} \eta_j \beta_j + \sum_{j=1}^{n_2} U_j C_j - \sum_{n_2+1}^{p_2} U_j C_j + \sum_{j=1}^{m_2} V_j D_j - \sum_{j=m_2+1}^{q_2} V_j D_j\right] > 0 \tag{1.7}$$

$$(iv)\ \Delta_2 = \left[\sum_{j=1}^{n_1} \xi_j A_j - \sum_{j=n_1+1}^{p_1} \xi_j A_j - \sum_{j=1}^{q_1} \eta_j B_j + \sum_{j=1}^{n_3} P_j E_j - \sum_{n_3+1}^{p_3} P_j E_j + \sum_{j=1}^{m_3} Q_j F_j - \sum_{m_3+1}^{q_3} Q_j F_j\right] > 0 \tag{1.8}$$

$$(v)\ |arg(z_1)| < \tfrac{1}{2}\, \Delta_1 \pi \ \text{ and } \ |arg(z_2)| < \tfrac{1}{2}\, \Delta_2 \pi \tag{1.9}$$

A detailed account of this function and its convergence conditions can be found in the paper [17].

2. Notations and Results used:

Throughout this paper D_x represents $\frac{d}{dx}$.

$$D_x^r\, f(x) = \frac{d^r}{dx^r}\, f(x) \tag{2.1}$$

$$(\, x\, D_x)^r\, f(x) = \left(x \frac{d}{dx}\right)^r \ f(x) \tag{2.2}$$

$$(D_x\, x)^r\, f(x) = \left(\frac{d}{dx}\, x\right)^r \ f(x) \tag{2.3}$$

where the operator $x\, D_x$ means the function of x in front of it is differentiated with respect to x and then multiplied by; $(\, x\, D_x)^r$ means that the operation by $x\, D_x$ is repeated r times;

$D_x\, x$ will mean that the function of x in front of it is first multiplied by x and then the product is differentiated with respect to x; D_x^r will mean that the operation by D_x is repeated r times.

In this paper for the sake of brevity we shall use the following contracted notation for I- function of two variables.

$$\bar{I}[z_1, z_2] = I^{\,0, n_1 : m_2, n_2 ;\, m_3, n_3}_{\,p_1, q_1 : p_2, q_2 ;\, p_3, q_3} \left[\begin{matrix} z_1 \\ z_2 \end{matrix} \middle| \begin{matrix} \mathcal{A} \,:\, \mathcal{C} ;\, \mathcal{E} \\ \mathcal{B} \,:\, \mathcal{D} ;\, \mathcal{F} \end{matrix} \right] \tag{2.4}$$

where

$\mathcal{A}$ stands for $(a_j;\, \alpha_j, A_j;\, \xi_j)_{1,p_1}$, $\mathcal{B}$ stands for $(b_j;\, \beta_j, B_j;\, \eta_j)_{1,q_1}$

$\mathcal{C}$ stands for $(c_j, C_j; U_j)_{1,p_2}$, $\mathcal{D}$ stands for $(d_j, D_j; V_j)_{1,q_2}$

$\mathcal{E}$ stands for $(e_j, E_j, P_j)_{1,p_3}$, $\mathcal{F}$ stands for $(f_j, F_j; Q_j)_{1,q_3}$

3. Main Results

In this section we have obtained the following three differentiation formulas.

Formula 1

$$D_x^r \left\{ x^\lambda\, I[\, z_1 x^{h_1}, z_2 x^{h_2}] \right\}$$

$$= x^{\lambda - r}\, I^{\,0, n_1+1 \,: m_2, n_2 ;\, m_3, n_3}_{\,p_1+1, q_1+1 : p_2, q_2 ;\, p_3, q_3} \left[\begin{matrix} z_1 x^{h_1} \\ z_2 x^{h_2} \end{matrix} \middle| \begin{matrix} (-\lambda;\, h_1, h_2; 1), \mathcal{A} & : & \mathcal{C}; & \mathcal{E} \\ \mathcal{B}, (\mathrm{r} - \lambda;\, h_1, h_2; 1) & : & \mathcal{D}; & \mathcal{F} \end{matrix} \right] \tag{3.1}$$

for $h_1 > 0,\ h_2 > 0$ and $\lambda \in \mathbb{C}$.

Formula 2

$$(x\, D_x - k_1) \ldots (x\, D_x - k_r) \left\{ x^\lambda\, I[\, z_1 x^{h_1}, z_2 x^{h_2}] \right\}$$

$$= x^{\lambda}\, I^{\,0, n_1+r \,: m_2, n_2 ;\, m_3, n_3}_{\,p_1+r, q_1+r : p_2, q_2 ;\, p_3, q_3} \left[\begin{matrix} z_1 x^{h_1} \\ z_2 x^{h_2} \end{matrix} \middle| \begin{matrix} (k_j - \lambda;\, h_1, h_2; 1)_{1,r}, \mathcal{A} & : & \mathcal{C}; & \mathcal{E} \\ \mathcal{B}, (1 + k_j - \lambda;\, h_1, h_2; 1)_{1,r} & : & \mathcal{D}; & \mathcal{F} \end{matrix} \right] \tag{3.2}$$

where $\lambda, k_j \in \mathbb{C}\ (j = 1, \ldots, k)$ and $h_1,\ h_2$ are real and positive.

Formula 3

$$(D_x\, x - k_1) \ldots (D_x\, x - k_r) \left\{ x^\lambda\, I[\, z_1 x^{h_1}, z_2 x^{h_2}] \right\}$$

$$= x^{\lambda}\, I^{\,0, n_1+r \,: m_2, n_2 ;\, m_3, n_3}_{\,p_1+r, q_1+r : p_2, q_2 ;\, p_3, q_3} \left[\begin{matrix} z_1 x^{h_1} \\ z_2 x^{h_2} \end{matrix} \middle| \begin{matrix} (k_j - \lambda - 1;\, h_1, h_2; 1)_{1,r}, \mathcal{A} & : & \mathcal{C}; & \mathcal{E} \\ \mathcal{B},\ (k_j - \lambda;\, h_1, h_2; 1)_{1,r} & : & \mathcal{D}; & \mathcal{F} \end{matrix} \right] \tag{3.3}$$

where $\lambda, k_j \in \mathbb{C}\ (j = 1, \ldots, k)$ and h_1, h_2 are real and positive.

Proofs:

To prove (3.1), express the left hand side of (3.1) using the contour integral (1.1), to obtain

$$L.H.S. = \left(\frac{d}{dx}\right)^r \left\{ x^\lambda \frac{1}{(2\pi i)^2} \int_{L_s} \int_{L_t} \phi(s,t)\theta_1(s)\theta_2(t)\, (z_1 x^{h_1})^s\, (z_2 x^{h_2})^t\, ds\, dt \right\} \tag{3.4}$$

where $\phi(s,t)$, $\theta_1(s)$ and $\theta_2(t)$ are given by (1.2), (1.3) and (1.4) respectively. Operating under the integral sign, the expression becomes

$$= \frac{1}{(2\pi i)^2} \int_{L_s} \int_{L_t} \{ \phi(s,t)\, \theta_1(s)\, \theta_2(t)\, z_1^s\, z_2^t$$

$$\times \prod_{j=0}^{r-1} (\lambda + h_1 s + h_2 t - j)\, x^{\lambda + h_1 s + h_2 t - r} \Big\} ds\, dt \tag{3.5}$$

Now it is easy to see that

$$\prod_{j=0}^{r-1} (\lambda + h_1 s + h_2 t - j\,) = \frac{\Gamma(1 + \lambda + h_1 s + h_2 t)}{\Gamma(1 + \lambda - r + h_1 s + h_2 t)} \tag{3.6}$$

Using (3.6) in (3.5) and interpreting the with the help of (1.1), the result follows.

To prove (3.2), express the left hand side using the contour integral (1.1), to obtain

$L.H.S. =$

$$\prod_{j=1}^{r} (x\, D_x - k_j) \left\{ x^\lambda \frac{1}{(2\pi i)^2} \int_{L_s} \int_{L_t} \phi(s,t)\theta_1(s)\theta_2(t)\, (z_1 x^{h_1})^s\, (z_2 x^{h_2})^t\, ds\, dt \right\} \tag{3.7}$$

where $\phi(s,t)$, $\theta_1(s)$ and $\theta_2(t)$ are given by (1.2), (1.3) and (1.4) respectively. Operating under the integral sign, the expression becomes

$$= \frac{1}{(2\pi i)^2} \int_{L_s} \int_{L_t} \{ \phi(s,t)\, \theta_1(s)\, \theta_2(t)\, z_1^s\, z_2^t$$

$$\times \prod_{j=1}^{r} (\lambda - k_j + h_1 s + h_2 t) x^{\lambda + h_1 s + h_2 t}\, ds\, dt \Big\} \tag{3.8}$$

writing $\prod_{j=1}^{r} (\lambda - k_j + h_1 s + h_2 t)$ as $\prod_{j=1}^{r} \frac{\Gamma((\lambda - k_j + 1 + h_1 s + h_2 t)}{\Gamma((\lambda - k_j + h_1 s + h_2 t)}$ and interpreting with the help of (1.1), the result follows.

The proof of (3.3) is same as that of (3.2).

4. Special cases

(i) When $k_1 = k_2 = \cdots = k_r = 0$ in (3.2) we get

$(x\, D_x)^r \left\{ x^\lambda\, I[\, z_1\, x^{h_1}, z_2 x^{h_2}] \right\}$

$$= x^{\lambda}\ I^{\,0\,,\,n_1+r\ \ :\,m_2,\,n_2\,;\ m_3,\,n_3}_{\,p_1+r\,,\,q_1+r\,:\,p_2\,,\,q_2\,;\,p_3\,,\,q_3}\left[\begin{matrix} z_1\,x^{h_1} \\ z_2\,x^{h_2}\end{matrix}\middle| \begin{matrix}(-\lambda\,;\ h_1,h_2\,;1)_{1,r}\,,\mathcal{A} & : & \mathcal{C}\,; & \mathcal{E} \\ \mathcal{B},(1-\lambda\,;\ h_1,h_2\,;1)_{1,r} & : & \mathcal{D}\,; & \mathcal{F}\end{matrix}\right] \tag{4.1}$$

for $h_1 > 0,\ h_2 > 0$ and $\lambda \in \mathbb{C}$.

(ii) When $k_1 = k_2 = \cdots = k_r = 0$ in (3.3) we get

$(D_x\, x)^r \left\{ x^{\lambda}\, I[\, z_1\, x^{h_1}\,, z_2 x^{h_2}]\, \right\}$

$$= x^{\lambda}\ I^{\,0\,,\,n_1+r\ \ :\,m_2,\,n_2\,;\ m_3,\,n_3}_{\,p_1+r\,,\,q_1+r\,:\,p_2\,,\,q_2\,;\,p_3\,,\,q_3}\left[\begin{matrix} z_1\,x^{h_1} \\ z_2\,x^{h_2}\end{matrix}\middle| \begin{matrix}(-\lambda-1\,;\ h_1,h_2\,;1)_{1,r}\,,\mathcal{A} & : & \mathcal{C}\,; & \mathcal{E} \\ \mathcal{B},\ (-\lambda\,;\ h_1,h_2\,;1)_{1,r} & : & \mathcal{D}\,; & \mathcal{F}\end{matrix}\right] \tag{4.2}$$

for $h_1 > 0,\ h_2 > 0$ and $\lambda \in \mathbb{C}$.

(iii) If we take $p_1 = q_1 = n_1 = 0$ and $z_2 \to 0$ in (3.1) to (3.3), we get the corresponding results involving I-functions established by Vyas and Rathie [19].

(iv) When all the exponents $\xi_j(j = 1, \dots, p_1)$, $\eta_j\ (j = 1, \dots, q_1)$, $U_j(j = 1, \dots, p_2)$, $V_j(j = 1, \dots, q_2)$, $P_j(j = 1, \dots, p_3)$, $Q_j(j = 1, \dots, q_3)$ of various gamma functions involved in (1.2), (1.3) and (1.4) are equal to unity, (3.1) to (3.3) gives the differentiation formulae for H - function of two variables defined by Mittal and Gupta [13]. Further by taking $p_1 = q_1 = n_1 = 0$, $z_2 \to 0$ and specializing the parameters, (3.1), (3.2) & (3.3) gives the differentiation formulas for H - function established by Gupta, K.C. et al. [6] and Nair [14] respectively.

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