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FREE VIBRATION ANALYSIS OF ISOTROPIC PLATE USING MULTIQUADRIC RADIAL BASIS FUNCTION

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Abstract: In this paper multiquadric radial basis function is used for free vibration analysis of isotropic plate. The spatial discretization of the differential equations generates greater number of algebraic equations than the unknown coefficients. The multiple linear regression analysis, which is based on the least square error norm, is employed to obtain the eigenvalues (λ) of natural frequencies of vibration for clamped and simply supported immovable rectangular plates. Numerical results obtained by this method are compared with those obtained by other analytical methods.
Key words: multiquadric radial basis function, spatial discretization, free vibration, rectangular plates.

Notations

a, b	Dimension of plates
D	Flexural rigidity of plates
E	Young's modulus
h	Thickness of plates
R	Aspect ratio (a/b)
ν	Poisson's ratio
ρ	Mass density of plates
w	Displacement in z direction

1. Introduction

Various numerical and analytical methods have been used for finding the natural frequencies of isotropic plate. But analytical methods are restricted to simple geometries and boundary conditions [1]. Today, widely used numerical methods for free vibration analysis of plates are finite difference and finite elements method. However, finite element method is often inefficient, as this requires large amount of time for generation of mesh. To avoid the mesh generation process for the analysis of isotropic plates, recently classes of new methods known as meshless methods are developed. These are element free Galerkin (EFG) method [2], natural neighbour Galerkin method [3], the method of finite sphere [4], dual reciprocity method (DRM)[5], particular integral method [6], multiple reciprocity method (MRM)[7-9]. In 1990 Kansa [10] developed the concept of solving partial differential equations using radial basis functions (RBFs). It is the correspondence between meshless scattered data interpolation and numerical solutions to partial differential equations. Franke [11] studied the evaluation of Radial Basis Function (RBF) for scattered data interpolation in terms of timing, accuracy and ease of implementation. Fedoseyev et al. [12] improved the accuracy of RBF method by placing the interior knots. Fasshauer and coworkers [13-14] used the multi-level methods with smoothing to improve the accuracy of RBF method. Chen et.al [15] studied the free vibration analysis of circular and rectangular plates by employing the RBF in the imaginary-part fundamental solution. Ferreira etal overcame ill conditioning that occurs in RBF method while solving free vibration analysis of laminates [16] by using preconditioning [17]. The present paper uses RBF function for free vibration analysis of rectangular isotropic plate and applies multiple linear regression analysis to overcome ill conditioning. This is collocation method; therefore more number of equations is generated than the number of unknowns. To overcome this incompatibility multiple linear regression analysis, which is based on the least square error norm, is employed to obtain the Eigen values (λ).

2. Governing differential equations for free vibration of isotropic plate

The governing equation for a free flexural vibration of a uniform thin plate in non-dimensional form is written as follow [15]:

$$\frac{1}{a^4}\left(\frac{\partial^4}{\partial x^4} + 2R^2\frac{\partial^4}{\partial x^2 \partial y^2} + R^4\frac{\partial^4}{\partial y^4}\right)w = \lambda^4 w(x, y) \tag{1}$$

$$\lambda^4 = \omega^2 \rho_0 h / D \tag{2}$$

$$\rho_0 = \rho h$$

Where w is displacement, ρ_0 is surface density, λ and ω are Eigen values and natural frequency, respectively. The flexural rigidity $D = Eh^3/12(1-\nu^2)$, where E is Young's modulus, ν is Poisson ratio and h is the plate thickness.

3. Governing differential equations for boundary conditions

- **Simply supported boundary conditions**

$$w = 0 \qquad \frac{\partial^2 w}{\partial n^2} = 0 \tag{3}$$

- **For all four edge clamped**

$$w = 0 \qquad \frac{\partial w}{\partial n} = 0 \tag{4}$$

4. The Eigen value problem

Consider a general differential equation

$$Aw = \lambda^4 w \quad \text{in } \Omega \tag{5}$$

$$Bw = 0 \quad \text{on } \partial\Omega \tag{6}$$

Where Ω is domain and $\partial\Omega$ is boundary of the domain. A is an arbitrary differential operator, B is an operator imposed as boundary conditions, such as Dirichlet, Neumann, and Robin. Let us denote $\{P_i = (x_i\ ,\ y_i)\}_{i=1}^{N}$ to be N collocation points in Ω *of* which $\{(x_i\ ,\ y_i)\}_{i=1}^{N_I}$ are interior points; $\{(x_i\ ,\ y_i)\}_{i=N_I+1}^{N}$ are boundary points. In multiquadric radial basis function method, it is assumed that the approximate solution for problem (5) can be expressed as

$$w(x, y) = \sum_{j=1}^{N} w_j \varphi_j(x, y) \tag{7}$$

Where $\{w_j\}_{j=1}^{N}$ are the unknown coefficients to be determined, and $\varphi_j(x_j, y_j)$ is a basis function. Most widely used radial basis functions are:

$\varphi(r) = r^3$ Cubic

$\varphi(r) = r^2 \log(r)$ Thin plate splines

$\varphi(r) = (1-r)^m{}_+ \, p(r)$ Wendland functions

$\varphi(r) = e^{-(cr)^2}$ Gaussian

$\varphi(r) = \sqrt{c^2 + r^2}$ Multiquadrics

$\varphi(r) = (c^2 + r^2)^{-1/2}$ Inverse multiquadrics

Here $r = \|P - P_j\|$ is the Euclidean norm between points $P = (x, y)$ and $P_j = (x_j, y_j)$. The Euclidian distance r is real and non-negative and c is a shape parameter, a positive constant.

5. Calculation of eigenvalues problem

Let N_B be the total points on the boundary $\partial\Omega$ and N_I be the total points inside the Ω and $N = N_I + N_B$. Applying the RBF in eqn (5), we get

$$\sum_{j=1}^{N} w_j A \varphi \|P_i - P_j\| = \lambda^4 \sum_{j=1}^{N} w_j \varphi \|P_i - P_j\| \quad (8)$$

$i = 1,2,............N_I$, $j = 1,2,............N$

$$L = [A\varphi(\|P_i - P_j\|)]_{N_I * N} \quad (9)$$

$$M = [\varphi(\|P_i - P_j\|)]_{N_I * N} \quad (10)$$

Applying RBF in eqn. (6), we get

$$\sum_{j=1}^{N} w_j B \varphi(\|P_i - P_j\|) = 0 \quad (11)$$

$i = N_{I+1},...................N$ and $j = 1,2,...................N$

$$N = B\varphi(\|P_i - P_j\|)_{N_B * N} \quad (12)$$

$$w^1 = [w_1 \;\; w_2 \;\; w_3 ... w_N]^T \quad (13)$$

Eqn (8) and (11) can be written as: $Lw^1 = \lambda^4 w^1 M$ (14)

$$Nw^1 = 0 \quad (15)$$

General Eigenvalue problem in the matrix form becomes

$$\begin{bmatrix} L \\ N \end{bmatrix} w^1 = \lambda^4 \begin{bmatrix} M \\ 0 \end{bmatrix} w^1 \quad (16)$$

The algorithm [16] of the standard eigenvalue problem has been used in the present analysis. Now, $L^1 w^2 = \lambda w^2$ (17)

$$L^1 = LD^{-1}\begin{bmatrix} I_{N_I * N_I} \\ 0_{N_B * N_I} \end{bmatrix} \tag{18}$$

Where, $w^2 = [w_1 \; w_2 \; w_3 ... w_{N_I}]^T$ (19)

$$D = \begin{bmatrix} M \\ N \end{bmatrix} \tag{20}$$

6. Multiquadric radial basis function for free vibration of isotropic plate

Substituting radial basis function in equation (1), we get

$$\frac{1}{a^4}\left(\sum_{j=1}^{N} w_j \frac{\partial^4}{\partial x^4}\varphi_j + 2R^2 \sum_{j=1}^{N} w_j \frac{\partial^4}{\partial x^2 y^2}\varphi_j + R^4 \sum_{j=1}^{N} w_j \frac{\partial^4}{\partial y^4}\varphi_j\right) = \lambda^4 \sum_{j=1}^{N} w_j \varphi_j \tag{23}$$

$$\frac{1}{a^4}\left(\frac{\partial^4}{\partial x^4}\varphi_j + 2R^2 \frac{\partial^4}{\partial x^2 y^2}\varphi_j + R^4 \frac{\partial^4}{\partial y^4}\varphi_j\right)\sum_{j=1}^{N} w_j = \lambda^4 \varphi_j \sum_{j=1}^{N} w_j \tag{24}$$

7. Boundary Conditions

- ***For Simple supported edge***

$$x = 0, a \qquad \sum_{j=1}^{N} w_j \varphi_j = 0 \tag{24}$$

$$y = 0, b \qquad \sum_{j=1}^{N} w_j \varphi_j = 0 \tag{25}$$

$$x = 0, a \qquad \sum_{j=1}^{N} w_j \frac{\partial^2}{\partial x^2}\varphi_j = 0 \tag{26}$$

$$y = 0, b \qquad \sum_{j=1}^{N} w_j \frac{\partial^2}{\partial y^2}\varphi_j = 0 \tag{27}$$

- ***For all four edge clamped***

$$x = 0, a \qquad \sum_{j=1}^{N} w_j \varphi_j = 0 \tag{28}$$

$$y = 0, b \qquad \sum_{j=1}^{N} w_j \varphi_j = 0 \tag{29}$$

$$x = 0, a \qquad \sum_{j=1}^{N} w_j \frac{\partial}{\partial x}\varphi_j = 0 \tag{30}$$

$$y = 0, b \qquad \sum_{j=1}^{N} w_j \frac{\partial}{\partial y}\varphi_j = 0 \tag{31}$$

Numerical examples

In this paper, free vibration analysis of isotropic plate is considered. The equations of the motion are solved in space domain using multiquadric radial basis function. Results are presented based on simple supported and clamped boundary conditions.

Case (1): Clamped isotropic plates

A clamped supported isotropic plate with dimensions 1.2 m x 0.9 m is considered. Present results of the Eigen values have been compared with Dickinson [18], Chen [15], Kang and Lee results with ANSYS software [19] and have been shown in Table 1. Results obtained by multiquadric radial basis function (MQRBF) method are in good agreement with those obtained by other method.

Table.1: Six eigenvalues of clamped supported rectangular isotropic plate

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
Dickinson [18]	5.964	7.730	9.151	9.975	10.30	11.99
Chen [15]	5.952	7.703	9.129	9.947	10.266	11.95
ANSYS (441 nodes)[19]	5.946	7.701	9.114	9.938	10.24	11.91
ANSYS (961 nodes) [19]	5.950	7.706	9.123	9.948	10.26	11.94
Kang and Lee [19]	5.952	7.703	9.131	9.955	10.27	11.95
Present method	5.954	7.703	9.132	9.953	10.29	11.97

Similarly, a circular plate with a radius (ρ=1 m) subjected to the clamped boundary condition is considered. Eigen values have been compared with other approaches Chen [15], and have been shown in Table 2. Results obtained by multiquadric radial basis function (MQRBF) method are in good agreement with those obtained by other method.

Table 2: The former six eigenvalues for the clamped circular plate using different approaches

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
NASA SP-160	3.196	4.611	5.906	6.306	7.144	7.799
Integral Eq.	3.2	4.6	5.9	6.3	7.2	7.9
Kang and Lee	3.196	4.611	5.906	6.306	7.144	7.799
Circulant method	3.20	4.61	5.91	6.31	7.14	7.80
Exact solution	3.196	4.611	5.906	6.306	7.144	7.799
Present method	3.197	4.613	5.906	6.307	7.146	7.801

Case (2): Simply supported isotropic plates

Whitney [20] has presented the lowest four natural frequencies of simple supported isotropic plates. Present results of the simply supported isotropic plates obtained by multiquadric radial basis function (MQRBF) method are in very close agreement to those of Whitney as seen in Table 3.

Table 3: Lowest four natural frequencies of simple supported rectangular isotropic plates, ($\omega = k\pi^2 / b^2 \sqrt{D/\rho}$)

Mode	m	n	$k[whitney]$	$k[presentmethod]$
1st	1	1	2.0	2.004
2nd	1	2	5.0	5.050
3rd	2	1	5.0	5.050
4th	2	2	8.0	8.039

Conclusions: Other numerical methods have some drawbacks, which have been explained above. Therefore, nowadays meshless methods are getting popularity. They have several advantages. Various meshless methods have been studied but in this paper multiquadric radial basis functions method has been applied to analyze the natural frequencies of simple supported and clamped isotropic plates.

Appendix I

Multiple regression analysis

$$A.a = p \tag{32}$$

where A is *(l*k)* coefficient matrix, a is (k*1) vector, *p* is *(l*1)* load vector.

Approximating the solution by introducing the error vector e, we get

$$p = Aa + e \tag{33}$$

where *e* is *(l*1)* vector.

To minimize the error norm, let us define a function S as

$$S(a) = e^T e = (p - Aa)^T (p - Aa) \tag{34}$$

The least-square norm must satisfy

$$(\partial S / \partial a)_a = -2A^T p + 2A^T Aa = 0 \tag{35}$$

This can be expressed as

$$a = (A^T A)^{-1} A^T P \tag{36}$$

$$or \quad a = B.P \tag{37}$$

The matrix *B* is evaluated once and stored for subsequent *usages.*

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