International Journal of Science, Environment and Technology, Vol. 3, No 3, 2014, 1246 – 1250
ISSN 2278-3687 (O)

NEWTON'S SECOND LAW IN GENERAL RELATIVITY

[1]Okpara, P.A., [2]Ekpe E.O. and [3]Oboma, D.N.
[1]Department of Industrial Mathematics and Applied Statistics
[2,3] Department of Industrial Physics
Ebonyi State University, Abakaliki
E-mail: paokapara@yahoo.com

Abstract: The weak equivalence principle (WEP) through the strong equivalence principle creates even in strongly curved spacetime some small regions of it for which geodesics of particles are not curved as the basis vector within such region are basically invariant and spacetime approximates to flat. We show that in such flat space inside a strongly curved space, the Einstein general relativity reduces to Newton's second law of motion $f^i = m\ddot{x}^i$.
Keywords: Newton's law, geodesic, curved spacetime, general relativity.

Introduction

In 1687, Sir Isaac Newton published the laws of motion which equilibrates inertial force due to acceleration of mass with the external force causing the acceleration. The second law which is the point of emphasis of this research states that the rate of change of state of motion of a body is directly proportional to the impressed motive force and acts in the direction of such a force. In strong gravitational fields, bodies move through curved trajectories in traveling through regions of varying gravitational intensities (Kleiss, 2010). However, if in some locally inertial frame for which gravity intensity is roughly constant, the Riemannian manifold reduces to the Euclidean space and trajectories of particles become straight lines. This is the Newtonian limit of the Einstein's general relativity. This deduction is in tune with the Einstein's postulate of the strong equivalence principle. It states that,

At every point in an arbitrary gravitational field, it is possible to choose a locally inertial system, such that (with a sufficiently small region around that point), the laws of nature take some form as an unaccelerated Cartesian coordinate system (Asaf, 2013). The dimension of such locally inertial system should be inversely related to the curvature of spacetime. Therefore, the action of gravity is attributed to the curvature of spacetime. The constancy of acceleration of falling bodies due to Galileo occur in this local inertial frame (Stuckey, 1993).

*Received Nov 04, 2013 * Published June 2, 2014 * www.ijset.net*

Theory

Mathematical tools are used to describe motion in spaces tangent to the curved manifold. These tools serve as building blocks for describing motion through the entire manifold. In curved manifold we transform a Cartesian coordinate system to a curvilinear coordinate system (Sokolinkoff, 1951). The basis vector of curvilinear coordinates describe displacements in the direction of each of these coordinates. To obtain the geometry of spacetime will require information on how basis vectors change along the manifold (Bertschinger, 1999). The vector displacement through a tangent plane for a general curvilinear coordinate system with Einstein summation convention is given by:

$$ds = e_i dq^i, \tag{1}$$

and

$$ds = e_j dq^j, \tag{2}$$

where dq^i, dq^j are contravariant components of the infitesimal displacements and e_j, e_i are the basis vectors (Riley, 2002). We use the inner product of equation (1) and equation (2) to define the line element such that

$$ds.ds = ds^2 = e_i dq^i . e_j dq^j,$$

$$= e_i . e_j dq^i dq^j$$

i.e.

$$ds^2 = (e_i . e_j) dq^i dq^j$$

$$= g_{ij} dq^i dq^j \tag{3}$$

where g_{ij}, a covariant tensor of the second rank, is the fundamental or metric tensor which determines the geometry of spacetime. For a vector $v = v^i e_i$, the covariant derivative is

$$\frac{dv}{dx^j} = \frac{dv^i}{dx^j} \; e_i + v^i \frac{de_i}{dx^j}$$

$$= \frac{dv^i}{dx^j} e_i + v^i T_{ij}^k e_k$$

where T_{ij}^k is the k-th component of the derivative. We are free to interchange dummy indices since their expansion will yield the same result.

$$\frac{dv}{dx^j} = \frac{dv^i}{dx^j} e_i + v^k T_{kj}^i e_i = \left(\frac{dv^i}{dx^j} + v^k T_{kj}^i \right) e_i$$

$$\frac{dv}{dx^j} = v_{;j}^i e_i \tag{4}$$

$v^i_{;j}$ is a convariant derivative component in the direction of e_i.

where T^i_{kj} is the Christoffel's symbols of the second kind. It is the i-th component of the vector $\frac{\partial e_k}{\partial x^j}$ that is

$$T^i_{kj} = \frac{\partial e_k}{\partial x^j} . e_i = T^i_{kj} e_i \tag{5}$$

Therefore. $\frac{\partial e_k}{\partial x^j} . e_i = T^i_{kj} e_i . e_i = T^i_{kj}$

To calculate the derivative of a vector along a curve $r(t)$ where t is a parameter, we consider the derivative of the vector v along the curve. For a coordinate system $u^i, i = 1,2,3$, the vector is written as

$\boldsymbol{r} = \sum_{i=1}^{3} v^i e_i$ which by Einstein's convention becomes,

$$\boldsymbol{r} = v^i e_i$$

so that

$$\frac{dr}{dt} = \frac{dv^i}{dt} e_i + v^i \frac{de_i}{dt}$$

$$= \frac{dv^i}{dt} e_i + v^i \frac{\partial e_i}{\partial x^k} \frac{dx^k}{dt}$$

$$= \frac{dv^i}{dt} e_i + v^i T^j_{ik} e_j \frac{dx^k}{dt}$$

Again, we swap dummy indices *i, j* to get that

$$\frac{dv}{dt} = \frac{dv^i}{dt} e_i + T^i_{jk} v^j \frac{dx^k}{dt} e_i$$

$$\frac{dv}{dt} = \left(\frac{dv^i}{dt} + v^j T^i_{jk} \frac{dx^k}{dt} \right) e_i \tag{6}$$

Equation (6) is the absolute derivative of the contravariant component v^i along the curve $r(t)$.

where $v^i_{;k} \frac{dx^k}{dt} = \frac{dv^i}{dt} + v^j T^i_{jk} \frac{dx^k}{dt}$

$$\frac{dv}{dt} = \frac{\delta v}{\delta t} e_i = \left(v^i_{;k} \frac{dx^k}{dt} \right) e_i \tag{7}$$

We recall that a geodesic in a flat space is a straight line which has two basic defining properties: (i) invariant direction of its vector which is always tangent to the curve and (ii) the separation between two locations is an extremum. Using the former, we seek for the intrinsic derivative of the tangent vector $\lambda = \lambda^i e_i$ and set it equal to zero, where $\lambda = \frac{dx^i}{dt}$.

then $\frac{\delta\lambda^i}{\delta t} = \frac{d}{dt}\left(\lambda^i e_i\right) = 0$

$$0 = \frac{d\lambda^i}{dt}e_i + \lambda^i \frac{\partial e_i}{\partial x^j}\frac{dx^j}{dt}$$

$$0 = \frac{d\lambda^i}{dt}e_i + \lambda^k T^i_{kj}\frac{dx^j}{dt}e_i$$

$$\left(\frac{d\lambda^i}{dt} + \lambda^k T^i_{kj}\frac{dx^j}{dt}\right)e_i = 0$$

$$\frac{\delta\lambda^i}{\delta t} = \lambda^i{}_{;j}e_i = 0$$

Since $e_i \neq 0, then \quad \lambda^i_{;j} = 0$

but $\quad \lambda^i = \frac{dx^i}{dt}$

Therefore, we substitute for $\lambda^i_{;}$

$$\frac{d}{dt}\left(\frac{dx^i}{dt}\right) + T^i_{kj}\frac{dx^k}{dt}\frac{dx^j}{dt} = 0$$

$$\frac{\delta\lambda^i}{\delta t} = \frac{d^2x^i}{dt^2} + T^i_{kj}\frac{dx^k}{dt}\frac{dx^j}{dt} = 0 \tag{8}$$

Equation (8) is the contravariant component of the intrinsic derivative of the tangent vector. It is the geodesic equation which for a parameter of time is the acceleration in curvilinear coordinates. Using this, we state the Newton's second law in contravariant form which becomes

$$f^i = ma = m\left(\ddot{x}^i + T^i_{kj}\dot{x}^k\dot{x}^j\right) = 0 \tag{9}$$

The Christoffels symbol here is connected to the metric tensor which defines the fundamental geometry of spacetime. Recall that

$$g_{ij} = e_i . e_j$$

Taking a cyclic permutation in *i,j,k*, we get that

$$\frac{<}{\partial x^i}\left(e_j . e_k\right) + \frac{\partial}{\partial x^j}\left(e_k . e_i\right) - \frac{\partial}{\partial x^k}\left(e_i . e_j\right) = 2T^l_{ij}e_{ik}$$

i.e $$\frac{1}{2}\left(\frac{\partial}{\partial x^i}\, g_{jk} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial}{\partial x^k}g_{ij}\right) = g_{ik}T^l_{ij} \tag{10}$$

We contract both sides with g^{mk} to get the levi-chivita connection

$g_{ik}\, g^{mk}\, T^l_{ij} = \delta^m_i T^l_{ij} = T^m_{ij}$

such that; $T^m_{ij} = \frac{1}{2}g^{mk}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial <_{ij}}{\partial x^k}\right)$ (11)

Equation (10) is the Christoffels symbols of the first kind while the connection coefficient is equation (11). From equation (11), in flat space where the metric tensor is invariant, the Christoffels symbols vanish because the metric tensor is constant and its differential is zero. Equation (9) reduces to

$$f^i = ma = m\ddot{x}^i$$

Conclusion

The geodesic equation in the external spacetime of a compact body is curved in traversing regions of varying gravitational intensities. This results to the curvature of spacetime. At far distances from the compact body for which curvature flattens, spacetime becomes flat and the fundamental tensor becomes constant. The generalized Riemannian manifold reduces to Euclidean space. However, even with a strongly curved spacetime a small region may exist where the gravitational intensity is fairly constant. Here, the metric tensor should remain invariant and the geodesics again should reduce to straight lines.

We therefore accommodate that the straightness of a geodesic which is a defining property of flat space identified with zero curvature also hold even in strongly curved spacetime only if the dimension of observation is such that the gravitational intensity is roughly constant. Then Newton's laws can be applied even inside a strongly curved spacetime.

REFERENCES

[1] Asaf Pe'er (2013): The Principle of equivalence and its consequences.

[2] Austen Berlet: Formulation of Einstein Field Equation through Newtonian spacetime.

[3] Edmund Bertschinger (1999): Introduction to Tensor Calculus for general Relativity, Massachusetts Institute of Technology.

[4] Louis Wonnell (2010): Schwarzschild Geometry of a Black Hole.

[5] Sokolinkoff, I.S. (1951). Tensor Analysis (Theory and Applications) Applied Mathematics Series. John Willey and Sons Inc. London. Chapman and Hall Limited.

[6] Stuckey, W.M. (1993): The Schwarzchild black hole as a gravitational mirrow. *AM. J. Phys.* 61(5) pp. 448-456.

[7] Ronald Kleiss (2010): The physics of spacetime, Radboud University, Nijmegen, the Netherland.

[8] Riley, K.F., Hobson, M.P. and Bence, S.J. (2002): Mathematical methods for Physics and Engineering. Cambridge Low – Price Editions.