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CYCLIC GRAPH - EDGE PRODUCT NUMBER

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Abstract: A graph G is said to be an edge product graph if there exists an edge function f: E → P such that the function f and its corresponding edge product function F on V satisfies that F(v) ∈ P for every v ∈ V and if $e_1, e_2, \ldots, e_p \in E$ such that $f(e_1) \cdot f(e_2) \cdot\ldots\cdot f(e_p) \in P$ then the edges $e_1, e_2, \ldots, e_p$ are incident on a vertex. In this paper, for a given cyclic graph G(V, E), the edge product number of cycles is found.
Keywords: edge function, edge product function, edge product graph, edge product number of a graph, cyclic graph.

1. Introduction

Most graph labeling methods trace their origin was introduced by Rosa [8]. A function f a ß-valuation of a graph G with q edges if f is an injection from the vertices of G to the set {1,2,3,..q}such that when each edge xy is assigned the label |f(x) - f(y)|, the resulting edge labels are distinct. A graph G(V,E) is called a sum graph if there exist a bijective labeling f from V to a set of positive integers S such that xy ∈ E if and only if f(x) + f(y) ∈ S.

Let s(G), the sum number of G, denote the minimum number of isolated vertices that must be added to G so that the resulting graph is a sum graph [6,7]. Ellingham [5] proved the conjecture of Harary [6] that s(T) = 1 for every tree $T \neq K_1$. In [2], the authors proved that $s(K_n) = (2n - 3)$. Sum graphs and sum number of graphs can be found more in [1].

The sum number of paths is found in [6]. Chen proved that trees obtained from a star by extending each edge to a path and trees all of whose vertices of degree not 2 are at least distance 4 apart are integral sum graphs [3,4]. Labeled graphs serve as useful models for a broad range of applications such as coding theory, x-ray crystallography, etc.

The authors have been studied some properties of edge function and the necessary conditions for a graph to be edge product function, with examples in [9]. The idea about edge product graphs and the edge analogue of product graphs are in [10]. Also we found the edge product number of graphs in paths. Another aspect of edge product labeling that caught the author's attention was that there were relatively few techniques and necessary conditions to show whether a graph is an edge product graph. In this paper, for a given cyclic graph G, we investigate their edge product number.

2. Previous Results

All graphs are finite and simple. Let G(V, E) be a graph. A bijection $f : E \to P$ where P is a set of positive integers is called an edge function of the graph G. Define $F(v) = \prod \{f(e): e \text{ is incident on } v\}$ on V. Then F is called the edge product function of the edge function f. G is said to be an edge product graph if there exists an edge function $f : E \to P$ such that f and its corresponding edge product function F on V satisfy the conditions that

(i) $F(v) \in P$ for every $v \in V$

(ii) If $e_1, e_2, \ldots, e_p \in E$ such that $f(e_1) \cdot f(e_2) \cdot \ldots \cdot f(e_p) \in P$, then $e_1, e_2, \ldots, e_p$ are incident on a vertex.

Let EPN(G), the edge product number of G,denote the minimum number K_2 components that must be added to G so that the resulting graph is an edge product graph. For any connected graph G other than K_2, $EPN(G) \geq 1$. Let EPN(G) = r. An edge function $f: E \to P$ and its corresponding edge product function F which make $G \cup rK_2$ an edge product graph are respectively called an optimal edge function and an optimal edge product function of G. Let $E = E_1 \cup E_2$ where E_1 is the edge set of G and E_2 that of rK_2. Then EPN(G) = Cardinality of the set $\{F(v): v \in V, F(v) \notin f(E_1)\}$. The function F is said to be an outer edge product function if $F(V) \cap f(E_1) = \Phi$ and inner edge product function if $F(V) \cap f(E_1) \neq \Phi$. The range of F has atleast r elements. It has exactly r elements if and only if F is outer edge product function.

3. Edge product number of cycles

Let C_q, for all $q \geq 3$ be a cycle on q vertices with $V_1 = \{v_1, v_2, \ldots, v_q\}$and $E_1 = \{v_i v_{(i+1)}: 1 \leq i \leq (q - 1)\} \cup \{v_q v_1\}$ be the edge set. The cycle C_q is triangle free and has no pendent vertex. Any optimal edge product function F of C_q, for $q \geq 4$, is an outer edge product function. In this section we investigate the edge product number of cycles.

Theorem 3.1. $EPN(C_q) = 2$ for some $q = 3$.

Proof: Consider C_3 is a cycle and r is a smallest number. If $EPN(C_3) = r$, then the graph $(C_3 \cup rK_2)$ is an edge product graph. The mapping f: $E \rightarrow P$ is an optimal edge function and F is an optimal edge product function of f. Let $\{v_1, v_2, v_3\}$ be the vertex set of the cycle C_3, then $F(v_1)$, $F(v_2)$ and $F(v_3)$ are the three distinct elements of P in which one of the function can be an edge of E. Therefore the number $r \geq 2$ and $EPN(C_3) = 2$. The following graph $C_3 \cup 2K_2$ shows that $EPN(C_3) = 2$.

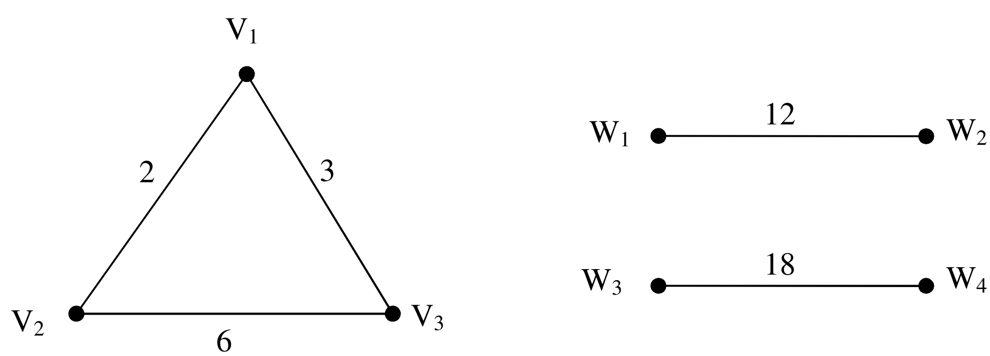

Figure 1

Theorem 3.2: $EPN(C_q) = 3$ for some $q \geq 4$.

Proof: Let C_q be a cycle for some $q \geq 4$. Since by theorem 3.1, we proved that $EPN(C_q) \geq 2$ for some $q \geq 4$. Then there arise two cases.

Case (1) when q is even

Let $q = 2p$ for some $p \geq 2$. Assume $EPN(C_{2p}) = 2$ for some p. Consider the graph $C_{2p} \cup 2K_2$. Let $V = \{v_1, v_2, \ldots, v_{2p}, w_1, w_2, w_3, w_4\}$ and $E = \{v_i v_{(i+1)}: 1 \leq i \leq (2p-1)\} \cup \{v_{2p}v_1, w_1w_2, w_3w_4\}$ of G. Let f: $E \rightarrow P = \{a_i: 1 \leq i \leq 2p\} \cup \{p_1, p_2\}$ be an optimal edge function and F be an optimal edge product function of f. The optimal edge function f is defined by $f(v_i v_{(i+1)}) = a_i$ for $1 \leq i \leq (2p-1)$; $f(v_{2p}v_1) = a_{2p}$ for $p \geq 2$; $f(w_1w_2) = p_1$ and $f(w_3w_4) = p_2$. Then the optimal edge product function F is defined by $F(v_1) = a_{2p} \times a_1$; $F(v_i) = (a_{(i-1)} \times a_i)$ for $2 \le i \le 2p$; $F(w_1) = F(w_2) = p_1$ and $F(w_3) = F(w_4) = p_2$. Then the range of the edge product function F has only two elements. For any two vertices u and v are adjacent, then we have $F(u) \neq F(v)$. Therefore, $F(v_1) = F(v_3) = \ldots = F(v_{2p-1})$ and $F(v_2) = F(v_4) = \ldots = F(v_{2p})$

(i.e), $a_{2p} \times a_1 = a_2 \times a_3 = \ldots = a_{(2p-2)} \times a_{(2p-1)}$ and $a_1 \times a_2 = a_3 \times a_4 = \ldots = a_{(2p-1)} \times a_{2p}$

(i.e), $(a_{2p} / a_2) = (a_2 / a_4) = \ldots = (a_{(2p-2)} / a_{2p})$. Suppose $(a_{(2p-2)} / a_{2p}) > 0$, then we obtain that $a_{2p} > a_2 > a_4 > \ldots > a_{(2p-2)} > a_{2p}$, which is a contradiction. Hence for some p, $EPN(C_{2p}) \ge 3$. By defining an appropriate edge function to prove $EPN(C_{2p}) = 3$.

Let $C_{2p} \cup 3K_2$ be the given graph. Let $V=\{v_1,v_2,\ldots,v_{2p},w_1,w_2,w_3,w_4,w_5,w_6\}$ and $E = \{v_iv_{(i+1)}: 1 \le i < (2p-1)\} \cup \{v_{2p}v_1, w_1w_2, w_3w_4, w_5w_6\}$ of G. Let $A=(3p + p) / 2$ and P be $\{2^{p+i}: 1 \le i \le p\} \cup \{2^{A+i}: 1 \le i \le p\} \cup \{2^{A+p+2}, 2^{A+2p+1}, 2^{A+2p+2}\}$.

Define the edge function $f: E \to P$ by $f(v_{(2i-1)}v_{2i}) = 2^{p+i}$ for $1 \le i \le p$

$f(v_{2i}v_{2i+1}) = 2^{A+p+1-i}$ for $1 \le i \le (p-1)$

$f(v_{2p}v_1) = 2^{A+1}$

$f(w_1w_2) = 2^{A+p+2}$

$f(w_3w_4) = 2^{A+2p+1}$ and $f(w_5w_6) = 2^{A+2p+2}$

Define the edge product function F of f by

$F(v_1) = f(v_{2p}v_1) \times f(v_1v_2) = 2^{A+1} \times 2^{p+1} = 2^{A+p+2} = f(w_1w_2)$

$F(v_{(2i+1)}) = f(v_{2i}v_{(2i+1)}) \times f(v_{(2i+1)}v_{(2i+2)})$ for $1 \le i \le (p-1)$

$= 2^{A+p+1-i} \times 2^{p+i+1} = 2^{A+2p+2} = f(w_5w_6)$

$F(v_{2i}) = f(v_{(2i-1)}v_{2i}) \times f(v_{2i}v_{(2i+1)})$ for $1 \le i \le (p-1)$

$= 2^{p+i} \times 2^{A+p-i+1} = 2^{A+2p+1} = f(w_3w_4)$

$F(v_{2p}) = f(v_{(2p-1)}v_{2p}) \times f(v_{2m}v_1) = 2^{2p} \times 2^{A+1} = 2^{A+2p+1} = f(w_3w_4)$

$F(w_1) = F(w_2) = f(w_1w_2) = 2^{A+p+2}$

$F(w_3) = F(w_4) = f(w_3w_4) = 2^{A+2p+1}$ and $F(w_5) = F(w_6) = f(w_5w_6) = 2^{A+2p+2}$.

Thus the range of the edge product function F has only three elements namely, 2^{A+p+2}, 2^{A+2p+1} and 2^{A+2p+2} and all are the elements of P. Hence the function F is into P. Now the elements of P in ascending order as: $2^{p+1}, 2^{p+2}, \ldots, 2^{2p}, 2^{A+1}, 2^{A+2}, \ldots, 2^{A+p}, 2^{A+p+2}, 2^{A+2p+1}$ and 2^{A+2p+2}. Then the elements of P satisfy the following three conditions:

(i) $2^{p+1} \times 2^{p+2} > 2^{2p}$

(ii) $2^{p+1} \times 2^{p+2} \times \ldots \times 2^{2p} = 2^{A} < 2^{A+1}$

(iii) $2^{p+1} \times 2^{A+1} = 2^{A+p+2}$

If the value of $f(e_1) \times f(e_2) \times \ldots \times f(e_r) = P$ where $e_1, e_2, \ldots, e_r \in E$ and $r > 1$ then P is among the three elements $2^{A+p+2}, 2^{A+2p+1}$ and 2^{A+2p+2}. The elements of P is divided into three sets, namely $P_1 = \{2^{p+1}, 2^{p+2}, \ldots, 2^{2p}\}$, $P_2 = \{2^{A+1}, 2^{A+2}, \ldots, 2^{A+p}\}$ and $P_3 = \{2^{A+p+2}, 2^{A+2p+1}, 2^{A+2p+2}\}$. Therefore $P = P_1 \cup P_2 \cup P_3$. Then the elements in the sets P_1, P_2 and P_3 have the following properties.

(i) $2^{A} < 2^{A+1}$

(ii) $(2^{p+1} \times 2^{p+2}) \times 2^{A+1} > 2^{A+2p+2}$

(iii) $2^{A+1} \times 2^{A+2} > 2^{A+2p+2}$

(iv) $2^{p+1} \times 2^{A+p+2} > 2^{A+2p+2}$

If the product of a collection of more than one element of P is in P then the collection consists of exactly two elements with one from P_1 and the other is from P_2 with the product being in P_3. The product of the smallest element in P_1 and the smallest in P_2 is equal to the smallest element in P_3. That is $2^{p+1} \times 2^{A+1} = 2^{A+p+2}$. The other products are greater than 2^{A+p+2}. The product of the smallest element in P_1 and the largest element in P_2 is equal to the largest element in P_3. That is, $2^{p+1} \times 2^{A+p} = 2^{A+2p+1}$. Therefore it is not possible to get any other product in P_3 with one element as 2^{p+i} and another from P_2. In both the cases the corresponding edges are incident on only one vertex.

The elements $2^{(A+2p+1)-(p+i)}$ and $2^{(A+2p+2)-(p+i)}$ for $2 \le i \le p$ are uniquely determined. Thus the collection of 2(p-1) elements gives the products $F(v_i)$ for $3 \le i \le 2p$. If $f(e_1) \times f(e_2) \times \ldots \times f(e_r) \in P$ for $r > 1$ and $e_1, e_2, \ldots, e_r \in E$ then $r = 2$ and the edges e_1 and e_2 are incident on a vertex.

Therefore for some $p \ge 2$, the graph $(C_{2p} \cup 3K_2)$ is an edge product graph. Hence $EPN(C_{2p}) \le 3$ for some $p \geq 2$ leads to the desired result that $EPN(C_{2p}) = 3$ for some $p \geq 2$. The following graph $C_{10} \cup 3K_2$ shows that $EPN(C_{10}) = 3$.

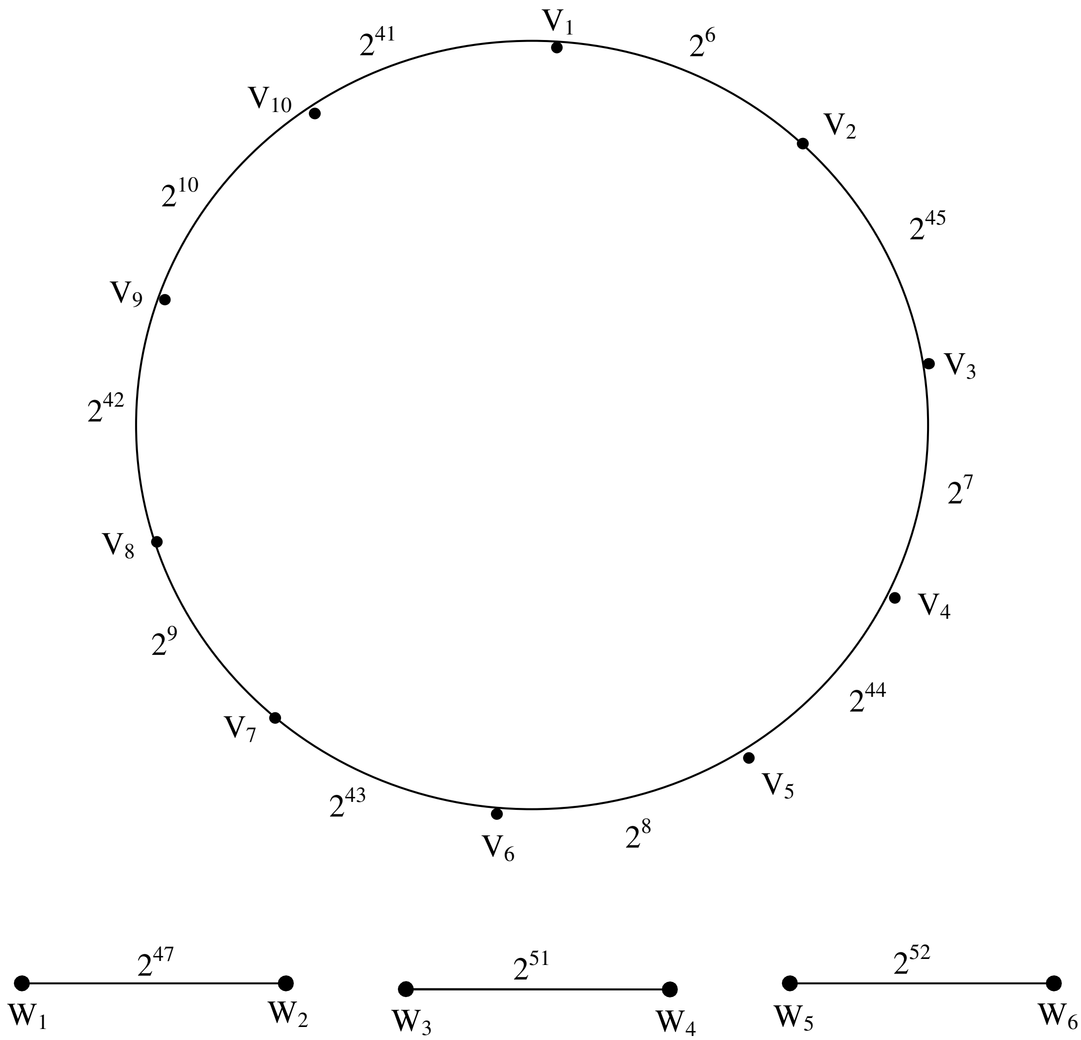

Figure 2

Case (2) when q is odd

Let $q = (2p + 1)$ for some $p \geq 2$. Suppose the graph $(C_{(2p+1)} \cup rK_2)$ and $EPN(C_{(2p+1)}) = r$ where r is the smallest number such that $C_{(2p+1)}$ is a cycle on $(2p + 1)$ vertices.

Then $(C_{(2p+1)} \cup rK_2)$ is an edge product graph. The vertex set and the edge set of $C_{(2p+1)}$ are $V_1 = \{v_1, v_2, \ldots, v_{2p}, v_{2p+1}\}$ and $E_1 = \{v_i v_{(i+1)}: 1 \leq i \leq 2p\} \cup \{v_{(2p+1)} v_1\}$ respectively.

Let f: E → P be an optimal edge function and F be its corresponding optimal edge product function. If the number r = 2, then the range of the function F has only two elements. $F(v_1) = F(v_3) = \ldots = F(v_{(2p-1)})$ and $F(v_2) = F(v_4) = \ldots = F(v_{2p})$. But the optimal edge product function $F(v_{(2p+1)})$ is neither $F(v_1)$ nor $F(v_{2p})$ as both the vertices v_1 and v_{2p} are adjacent to $v_{(2p+1)}$. Hence r ≥ 3 and $EPN(C_{(2p+1)}) \geq 3$ for all p ≥ 2. Now by giving an appropriate edge function to show that $EPN(C_{(2p+1)}) = 3$ for all p ≥ 2. Consider the graph $(C_{(2p+1)} \cup 3K_2)$ and V = $\{v_1, v_2, \ldots, v_{(2p+1)}, w_1, w_2, w_3, w_4, w_5, w_6\}$, the vertex set and E = $\{v_iv_{(i+1)}: 1 \leq i \leq 2p\} \cup \{v_{(2p+1)}v_1, w_1w_2, w_3w_4, w_5w_6\}$, the edge set of G. The elements of P is $\{2^{a-1+i}: 1 \leq i \leq p\} \cup \{2^{b+p-1-i}: 1 \leq i \leq (p-1)\} \cup \{2^{(a+b)/2}, 2^{((a+b)/2)+(p-1)}\} \cup \{2^{a+((a+b)/2))+(p-1)}, 2^{a+b+p-2}, 2^{a+b+p-1}\}$ where a = (p + 2) and b = $(3p^2 + 8p + 6)$.

Define the edge function f: E → P by $f(v_{(2i-1)}v_{2i}) = 2^{a-1+i}$ for $1 \leq i \leq p$; $f(v_{2i}v_{(2i+1)}) = 2^{b+p-1-i}$ for $1 \leq i \leq (p-1)$; $f(v_{2p}v_{(2p+1)}) = 2^{(a+b)/2}$; $f(v_{(2p+1)}v_1) = 2^{((a+b)/2)+(p-1)}$; $f(w_1w_2) = 2^{a+((a+b)/2))+(p-1)}$; $f(w_3w_4) = 2^{a+b+p-2}$ and $f(w_5w_6) = 2^{a+b+p-1}$.

Thus for the edge function f, the graph $(C_{(2p+1)} \cup 3K_2)$ is an edge product graph for all p ≥ 3. Hence $EPN(C_{(2p+1)}) = 3$ for all p ≥ 3. Thus we obtain that $EPN(C_q) = 3$ for all q ≥ 4. The following graph $C_5 \cup 3K_2$ shows that $EPN(C_5) = 3$.

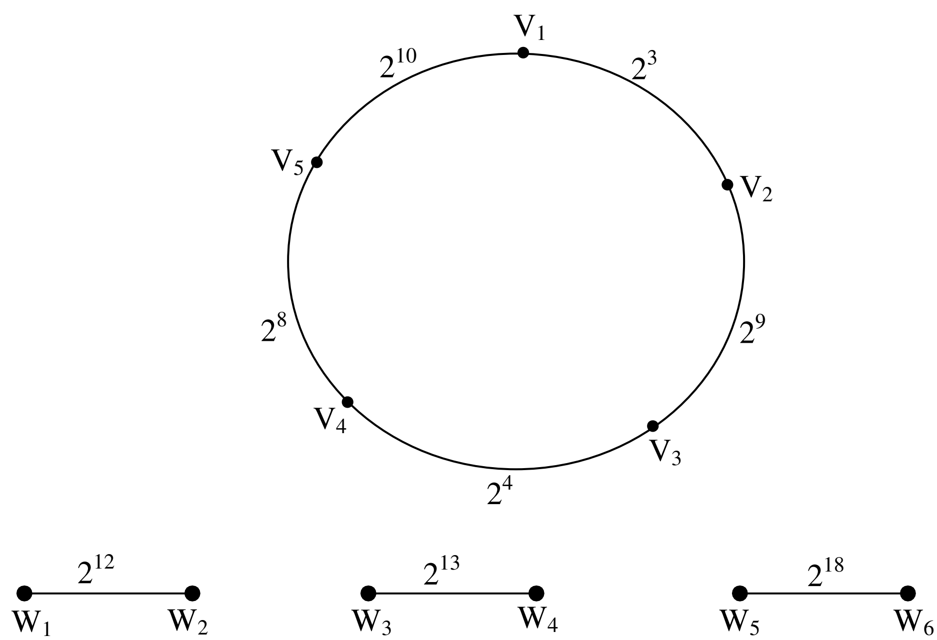

Figure 3

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