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ON EXACT SOLUTIONS OF (2+1)–DISSIPATIVE ZK EQUATION

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Abstract: The exact solutions of the (2+1)-dimensional Zabolotskaya--Khokhlov equation with dissipative terms are found by using the similarity transformation method. The solutions are analyzed physically through their evolutional profiles.
Keywords: Zabolotskaya-Khokhlov equation. Similarity transformations method, invariant solutions.

1. INTRODUCTION

The nonlinear partial differential equations (NPDEs) have extensive physical applications to describe complex phenomena in fluid dynamics, biology and nonlinear optics as described, e.g., in Refs. [1-5]. Therefore, seeking the exact solutions of NPDE's became very important and significant in the fields of nonlinear sciences.

As a suitable illustration, authors considered the following form of Zabolotskaya-Khokhlov (ZK) equation with dissipative terms

$$\mathbf{u_{xt}} + (\boldsymbol{u_x})^2 + \boldsymbol{u}\,\mathbf{u_{xx}} + \boldsymbol{\lambda}\,\mathbf{u_{xxx}} + \boldsymbol{\mu}\,\mathbf{u_{yy}} = \mathbf{0}, \qquad \mathbf{(1.1)}$$

where $u = u(x, y, t)$ is used for brevity and λ and μ are non zero real constants (Responsible for dissipation). Throughout the article, the term equipped with subscript denote the partial derivative of the indicated function with respect to the subscript variable.

The ZK Eq. (1.1) can be derived from the (2+1)-dimensional Burgers' equation under some parametric restriction [1,3,6]. In the past few decades, a great efforts had been done to solve the ZK equation by a diverse group of scientists, notably ZK equation is an approximation to describe the propagation of a confined three-dimensional beam in a slightly nonlinear medium without dispersion or absorption [1].

Infinitesimal symmetries and exact solutions of the ZK equation with a specific form had been found in the Refs. [7,8]. The propagation of sound wave in a nonlinear medium is well described [9]. The general derivation of the ZK equation was also established [7,8]. The

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solutions of obliquely interacting N travelling waves to the ZK equation were obtained [11]. Some similarity reductions and the exact solutions of the ZK equation (1.1) are found in [12]. Later on, the Fréchet derivatives were used to generate the Lie symmetries of the ZK equation and got its some exact solutions [6]. The travelling wave-like solutions of the ZK equation with variable coefficients were obtained by transforming it into the Riccati equation [13]. Motivated by the rich treasure of the ZK Eq. (1.1) in the available literature [1-13], authors attained the new invariant solution of the ZK equation by using the similarity transformations method (STM).

The purpose of this work is to generate Lie symmetries of the ZK equation and then to get new exact solutions by using the STM. The STM is a widely applied and quite successful technique for finding the closed form solutions of nonlinear partial differential equations (PDE's) or ordinary differential equations (ODE's). The STM is based on the study of Lie transformation groups that leave differential equations invariant. Indeed, the invariance property enables us to reduce the number of variables by one. Thus, using this procedure repeatedly and with the help of similarity variables, one can reduce the ZK equation to an ODE which occurs nonlinear in general. In some cases, it may be possible to solve this ODE analytically or some time the ODE must be solved numerically using some boundary conditions. The references [7,8,12,14-24] can be profitably consulted to study the required theory for the description of the STM and its applications.

2. INVARIANT SOLUTIONS

In this section, authors described the methodology for STM. To save the space, author touched every step briefly, but this will whet the one's appetite for delving into the same. First, authors generated the Lie symmetries of the ZK Eq. (1.1) and then found its invariant solutions. Therefore, one can exploit the one-parameter (ε) Lie group of point transformations of Eq. (1.1) as follows.

$$\left.\begin{aligned}
x^* &= x + \varepsilon\, \xi^1(x,y,t,u) + O(\varepsilon^2),\\
y^* &= y + \varepsilon\, \xi^2(x,y,t,u) + O(\varepsilon^2),\\
t^* &= t + \varepsilon\, \tau(x,y,t,u) + O(\varepsilon^2),\\
u^* &= u + \varepsilon\, \eta(x,y,t,u) + O(\varepsilon^2),\\
{u^*}_{x^*} &= \theta_x + \varepsilon\, \eta_x + O(\varepsilon^2),\\
{u^*}_{x^*x^*} &= \theta_{xx} + \varepsilon\, \eta_{xx} + O(\varepsilon^2),\\
{u^*}_{x^*t^*} &= \theta_{xt} + \varepsilon\, \eta_{xt} + O(\varepsilon^2), \text{etc}
\end{aligned}\right\} \qquad \textbf{(2.1)}$$

where ξ^1, ξ^2, τ and η are the infinitesimals for the variable x, y, t and u respectively, and $\theta\ (x, y, t)$ is the solution of the ZK Eq. (1.1).

Further, we assume Eq. (1.1) is invariant. Thus, the surface invariant condition on the ZK Eq. (1.1) provides

$$[\eta_{xt}] + 2[\eta_x]\theta_x + \eta\,\theta_{xx} + u\,[\eta_{xx}] + \lambda\,\eta_{xxx} + \mu\,\eta_{yy} = 0, \qquad \textbf{(2.2)}$$

where $[\eta_x]$, $[\eta_{xx}]$ and $[\eta_{xx}]$ etc. stand for prolongations [14,17].

Substitution of the values of prolongations from [14,17] into (2.2) provides an equation in terms of the partial derivatives of $\xi^{(1)}, \xi^{(2)}, \tau$ and η. So we can arrive on the following system of coupled PDE's by equating to zero the coefficients of various monomials.

$$\left.\begin{aligned} &\xi^{(1)}_u = \xi^{(2)}_u = \xi^{(2)}_x = \tau_x = \tau_y = \tau_u = 0,\\ &\xi^{(2)}_{yt} = \xi^{(2)}_{yy} = 0,\\ &\xi^{(1)}_x = \frac{2}{3}\xi^{(2)}_y,\ \xi^{(1)}_y = -\frac{1}{2\mu}\xi^{(2)}_t,\\ &\tau_t = \frac{4}{3}\xi^{(2)}_y,\ \eta = \xi^{(1)}_t - \frac{2u}{3}\xi^{(2)}_y. \end{aligned}\right\} \qquad \textbf{(2.3)}$$

Solving the system of PDE's (2.3) for infinitesimals ξ^1, ξ^2, τ and η one can find

$$\left.\begin{aligned} &\xi^{(1)} = \mathbf{a_1 x} - \frac{y}{2\mu}F_1'(t) + F_2(t),\\ &\xi^{(2)} = \frac{3}{2}\mathbf{a_1 y} + F_1(t),\\ &\tau = 2\mathbf{a_1 t} + a_2,\\ &\eta = -\mathbf{a_1 u} - \frac{y}{2\mu}F_1''(t) + F_2{}'(t), \end{aligned}\right\} \qquad \textbf{(2.4)}$$

where $\mathbf{a_1}$ and a_2 are the arbitrary constants and $F_1(t)$ and $F_2(t)$ are the arbitrary functions. The similar values of infinitesimals (2.4) for (1.1) are also generated in [6, 12]. The prime denotes the differentiation of a function with respect to its indicated variable throughout the article. Further, the choices of $F_1(t)$ and $F_2(t)$ provide us the new physically meaningful solutions of the Eq. (1.1). Therefore, authors considered $F_1(t) = -2\mu a_3 t + a_4$ and $F_2(t)= a_5 t + a_6$ with arbitrary constants a_3, a_4, a_4, and a_5. Therefore, Eq. (2.4) recasts

$$\left.\begin{aligned} &\xi^{(1)} = \mathbf{a_1 x} + a_3 y + a_5 t + a_6,\\ &\xi^{(2)} = \frac{3}{2}\mathbf{a_1 y} - 2\mu a_3 t + a_4,\\ &\tau = 2\mathbf{a_1 t} + a_2,\\ &\eta = -\mathbf{a_1 u} + a_5. \end{aligned}\right\} \qquad \textbf{(2.5)}$$

Therefore, the symmetry Lie algebra L_5 of (1.1) is generated by the operators

$$\mathbf{V_1 = x\frac{\partial}{\partial x} + \frac{3}{2}y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}},$$

$$\mathbf{V_2 = \frac{\partial}{\partial t}}, \quad \mathbf{V_3 = y\frac{\partial}{\partial x} - 2\mu t\frac{\partial}{\partial y}}, \quad \mathbf{V_4 = \frac{\partial}{\partial y}}, \quad \mathbf{V_5 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}}, \quad \mathbf{V_6 = \frac{\partial}{\partial x}}.$$

The L^5 is the direct sum of $V_1, V_2, \ldots, V_6$ and its commutator table is given by the table-1. The $(i, j)^{th}$ entry of the table represents the Lie bracket $[V_i\ V_j] = V_iV_j - V_jV_i$. One can observe that the table is skew-symmetric with zero diagonal elements.

Table -1: Commutator Table

*	V_1	V_2	V_3	V_4	V_5	V_6
V_1	0	$-2V_2$	$\frac{1}{2}V_3$	$-\frac{3}{2}V_4$	V_5	$-V_6$
V_1	$2V_2$	0	$-2\mu V_4$	0	V_6	0
V_1	$-\frac{1}{2}V_3$	$2\mu V_4$	0	V_6	0	0
V_1	$\frac{3}{2}V_4$	0	V_6	0	0	0
V_1	$-V_5$	$-V_6$	0	0	0	0
V_1	V_6	0	0	0	0	0

Table-1 shows that the generators V_1, V_2,... V_6 are linearly independent. Thus, to get the invariant solutions of the ZK Eq. (1.1), the characteristic equations are

$$\frac{dx}{\xi^{(1)}(x,y,t,u)} = \frac{dy}{\xi^{(2)}(x,y,t,u)} = \frac{dt}{\tau(x,y,t,u)} = \frac{du}{\eta(x,y,t,u)}. \quad \textbf{(2.6)}$$

The different forms of the solutions of ZK Eq. (1.1) are obtained by assigning the particular values to $a_i's$ $(1 \le i \le 6)$. Therefore, the STM predicts the following cases to generate the different forms of the exact solutions.

Case (I): If $a_1 \neq 0$, then comprising (2.5) and (2.6) yields

$$\frac{dx}{x+Ay+Bt+C} = \frac{dy}{\frac{3}{2}y-2\mu At+D} = \frac{dt}{2t+E} = -\frac{du}{u-B}, \quad \textbf{(2.7)}$$

where $A = \frac{a_3}{a_1}$, $B = \frac{a_5}{a_1}$, $C = \frac{a_6}{a_1}$, $D = \frac{a_4}{a_1}$ and $E = \frac{a_2}{a_1}$. The similarity form of the solution of the Eq. (1.1) can be written as

$$\boldsymbol{u = B + \frac{1}{\sqrt{T}}F(X,Y)}, \quad \textbf{(2.8)}$$

with similarity variables

$$X = \frac{x + b_3 t + b_4 y + b_5}{\sqrt{T}} \text{ and } Y = \frac{y+at+b}{T^{3/4}} \quad \textbf{(2.9)}$$

where $T = 2t + E$, $a = 4\mu A$, $b = \frac{8}{3}\mu AE + \frac{2D}{3}$,

$b_3 = -(B + Aa)$, $b_4 = -2A$ and

$b_5 = C - 3Ab + (Aa - B)E.$

Inserting the value of u from the equation (2.8) into (1.1), we get the following equation.

$$-2F_X - XF_{XX} - \frac{3}{2}Y F_{XY} + (F_X)^2 + F F_{XX} + \lambda F_{3X} + \mu F_{YY} = 0. \quad \textbf{(2.10)}$$

One can find a new set of the following infinitesimals by applying the STM on (2.10).

$$\overline{\xi^{(1)}} = -\frac{6\alpha}{8\mu}Y + \beta, \overline{\xi^{(2)}} = \alpha, \overline{\eta} = \frac{3\alpha}{8\mu}Y + \beta, \quad \textbf{(2.11)}$$

where α and β are arbitrary parameters. It follows that the characteristic equation for (2.10) is given by

$$\frac{dX}{-\frac{6\alpha}{8\mu}Y+\beta} = \frac{dY}{\alpha} = \frac{dF}{\frac{3\alpha}{8\mu}Y+\beta}. \quad \textbf{(2.12)}$$

Now, the following sub cases (IA) and (IB) are arisen:
Case (IA): If $\alpha \neq 0$, then the STM constructs the following form of F for the Eq. (2.10).

$$2F + X - \frac{3\beta}{\alpha}Y = \phi(X_1), \quad \textbf{(2.13)}$$

where ϕ is the similarity function and $X_1 = X + \frac{3}{8\mu}Y^2 - \frac{\beta}{\alpha}Y$ is the similarity variable. Substitution of F from (2.13) into (2.10) provides the following ODE.

$$2\lambda\phi''' + \phi\phi'' + \phi'^2 + \left(2\frac{\mu\beta^2}{\alpha^2} - 3X_1\right)\phi'' + 6\left(\frac{1}{4\mu} - 1\right)\phi' + 3 = 0. \quad \textbf{(2.14)}$$

Its primitive is given by the following ODE under the condition $\mu = 1/2$ otherwise can be solved numerically.

$$2\lambda\phi'' + \phi\phi' + \left(\frac{\beta^2}{\alpha^2} - 3X_1\right)\phi' + 3X_1 = c_1, \quad \textbf{(2.15)}$$

where c_1 is an arbitrary constant of integration.

Case (IB): If $\alpha = 0$, then applying the same procedure as we have done in the case (IA). Therefore, one can have the following similarity form of the function F for the Eq. (2.10)

$$F = X + \phi_1(Y), \quad \textbf{(2.16)}$$

with similarity variable Y and similarity function $\phi_1(Y)$.

Substituting F from (2.16) into (2.10) provides

$$\phi_1(Y) = \frac{Y^2}{2\alpha} + \alpha_1 Y + \beta_1, \quad \textbf{(2.17)}$$

where α_1 and β_1 are arbitrary constants of integration.

Comprising Eqs. (2.8), (2.16) and (2.17), solution of the ZK equation (1.1) is furnished by

$$u(x,y,t) = B + \frac{x+b_3t+b_4y+b_5}{\sqrt{2t+E}} + \frac{1}{2\mu}\frac{(y+at+b)^2}{(2t+E)^2} + \alpha_1\frac{(y+at+b)^2}{(2t+E)^{5/4}} + \frac{\beta_1}{(2t+E)^{1/2}} \qquad \textbf{(2.18)}$$

3. ANALYSIS AND DISCUSSION

This section deals with the physical analysis of the exact solution (2.18) of ZK equation (1.1). Solution is verified by *maple 13*. As the solution involves arbitrary constants $\alpha 1$, $\alpha 2$, $\beta 1$, $\beta 2$ and A, $B1$, C etc., hence the solution leads to the physically meaningful value of $u(x, y, t)$. Author presented the solution graphically via Fig 1. In the figure, $\mu = 1.5497$ is taken. The solution profile is conserved since $u \to$ constant (A) as $t \to \infty$. After $t = 5$, it becomes steady for this choice of parameter.

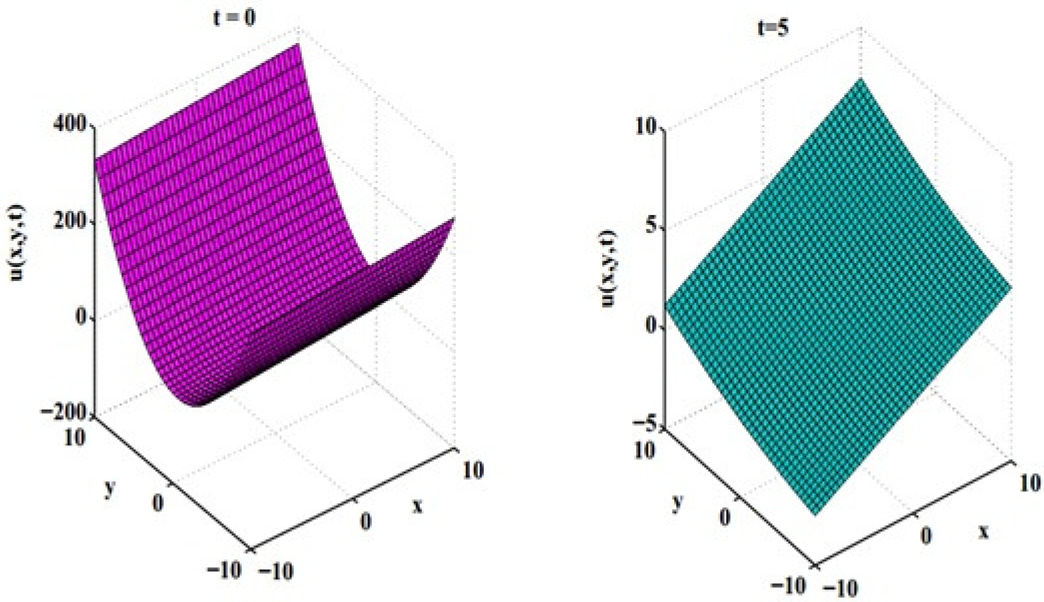

Figure 1: **Profile of (2.18) showing that the nonlinear behavior of wave becomes steady**

4. CONCLUSION

Some new explicit and exact solution of the ZK equation has been found by using the STM successively. The STM predicts new forms of $u(x, y, t)$. So these forms generated the new semi analytic solution. The dissipation factor μ plays an important role providing the solution of physical importance like in ultrasound transducer and tumour ablation techniques. The solution can serve as benchmark in the accuracy testing and comparison of numerical solutions. This work will illuminate the new research if someone is able to find the solutions of (2.14) and (2.15).

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REFERENCES

[1] E.A. Zabolotskaya, and R.V. Khokhlov, Sov. Phys. Acoust. **15** (1969) 35.

[2] S.H. Timothy, and M.F. Hamilton, J. Acoust. Soc. Am. **84** (1988) 1488.

[3] V.P. Kuznetsov, Sov. Phys. Acoust. **16** (1971) 467.

[4] P. Hariharan, M.R. Myers, and R.K. Banerjee, Phys. Med. Biol. **52** (2007) 3493.

[5] A. Pedersen, M. Vestrheim, and P. Lunde, published in the proceedings of the international conference "Underwater Acoustic Measurements: Technologies & Results," Heraklion, Crete, Greece, 28th June-1st July 2005.

[6] M.H.M. Moussa, and R.M. El Shikh, Physica A **371** (2006) 325.

[7] W. Fushchich, W.M. Shtelen, and N.I. Serov, *Mathematics and Its Applications: Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics,* Dordrecht, Kluwer Academic Publishers (1993).

[8] N. Euler, and W–H. Steeb, *Continuous Symmetries, Lie Algebras and Differential Equations,* Bibliographisches Institut, Mannheim, Germany (1992).

[9] M.J. Ablowitz, A. Ramani, and H. Segur, J. Math. Phys. **21** (1980) 715.

[10] T. Taniuti, Wave motion **12** (1990) 373.

[11] Y. Murakami, J. Phys. Soc. Jpn. **59** (1990) 1.

[12] M. Tajiri, INMP **2** (1995) 392.

[13] Y.Z. Peng, E.V. Krishnan, and H. Feng, Pramana–J. Phys. **71** (2008) 49.

[14] G.W. Bluman, and J.D. Cole, *Similarity Methods for Differential Equations,* Springer Verlag, New York (1974), p. 143–166.

[15] L.V. Ovsiannikov, *Group Analysis of Differential Equations,* Academic Press, New York (1982).

[16] W.F. Ames, *Nonlinear Partial Differential Equations in Engineering*, vol. 2, Academic Press, New York (1989).

[17] P.J. Olver, *Applications of Lie Groups to Differential Equations*: Graduate Texts in Mathematics, Springer Verlag, New York (1993) p. 30–130.

[18] M. Kumar, and Y.K. Gupta, Pramana–J Phys. **74** (2010) 883.

[19] H.Y. Zhi, Commun. Theor. Phys. (Beijing, China) **59** (2013) 263.

[20] Xu-Xia Song, Commun. Theor. Phys. (Beijing, China) **59** (2013) 253.

[21] M. Nadjafikhah, and F. Ahangari, Commun. Theor. Phys. (Beijing, China) **59** (2013) 335.

[22] M.H.M. Moussa, IJNS. **39** (2001) 1565.

[23] X.Y. Tang, S.Y. Lou, and Y. Zhang, Phys. Rev. E **66** (2002) 046601.

[24] D.S. Wang, Y.J. Ren, and H.Q. Zhang, Appl. Math. E–Notes **5** (2005) 157.

[25] R. Kumar, M. Kumar, and A. Kumar, IOSR–JM **6** (2013) 23.

[26] B. Bira, and T. Raja Sekhar, Exact solutions to magnetogasdynamics using Lie point symmetries, Meccanica **48** (2013) 1023.

[27] M. Kumar, and R. Kumar, Meccanica **49** (2014) 335.

[28] S. Lie, Arch. Math. **6** (1881) 328.

[29] W. Hereman, *Symbolic software for Lie symmetry analysis, in: N.H. Ibragimov (Ed.), CRCHandbook of Lie Group Analysis of Differential Equations: New Trends in Theoretical Developments and Computational Methods*, vol. 3, CRC Press, Boca Raton, Florida (1996) p. 367–413 (Chapter 13).

[30] E. Yasar, and T. Özer, Comput. Math. Appl. **59** (2010) 3203.