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A SPECIAL TYPE OF APPROXIMATE SOLUTIONS FOR CERTAIN NONLINEAR POTENTIAL PDES IN HYPERGEOMETRIC FORM

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Abstract: In the present paper a special type of approximate solution is obtained for Potential KdV equation and Potential KdV-Burger equation by employing a regular perturbation method. The perturbation series emerge in the form of geometric and hyper geometric series respectively. By introducing parameters suitably as coefficients in the Potential KdV equations, shockwave solution is obtained in the limiting case.
Key words: Potential KdV Equation, Potential KdV-Burger equation, regular perturbation series, Hyper geometric series, Shock wave.

INTRODUCTION

Burger equation and KdV equation are two classical and well-known among leading integrable nonlinear partial differential equations [1]. They posses exact solution in solitary wave form which exhibit shockwave in limiting case ([2], [4], [5]). In the present paper a special type of exact solution is obtained for Potential KdV equation and a approximate solution is obtained for Potential KdV-Burger equation by employing a regular perturbation method. By introducing parameters suitably as coefficient in the equations, shockwave solution is obtained for KdV equation in the limiting case. The perturbation series for potential KdV equation is geometric in nature and its formal sum provides exact solution. The perturbation series for potential KdV-Burger equation is in the form of Standard Gauss hyper geometric series: ${}_2F_1(a,b;c;z)$ [3].

POTENTIAL KdV EQUATION

The KdV equation in the standard form is

$$u_t + 6uu_x + \sigma u_{xxx} = 0 \tag{1}$$

If we substitute $u = -V_x$ then the resulting equation is called potential KdV equation, given by

$$V_t - 3V_x^{\ 2} + \sigma V_{xxx} = 0 \tag{2}$$

First let us scale by perturbation parameter

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$$V = \varepsilon U \tag{3}$$

Substitution of (3) in (2) will lead to a perturbed potential KdV equation

$$U_t + \sigma U_{xxx} = 3\varepsilon U_x^{\ 2} \tag{4}$$

Next, by seeking solution of (4) in the form of a perturbation series

$$U(x,t,\varepsilon) = U_0(x,t) + \varepsilon U_1(x,t) + ... + \varepsilon^n U_n(x,t) + ...$$

one can break the equation (4) into following hierarchy of linear equations:

$$U_{0,t} + \sigma U_{0,xxx} = 0$$

$$U_{n,t} + \sigma U_{n,xxx} = 3\sum_{j=0}^{n-1} U_{n-1-j,x} U_{j,x}$$

$$n = 1, 2, 3, \ldots$$

Zero-order term:

Let us choose a particular traveling wave solution in the exponential form given by

$$U_0 = \sqrt{\sigma} e^{\frac{-1}{\sqrt{\sigma}}(x-t)} , \qquad \sigma > 0$$

First-order term:

Let us choose $U_1(x,t) = a_1 e^{\frac{-2}{\sqrt{\sigma}}(x-t)}$ and

Substitute in $U_{1,t} + \sigma U_{1,xxx} = 3U_{0,x}^{\ 2}$

Then $a_1\left[\frac{2}{\sqrt{\sigma}} + \sigma\left(\frac{-8}{\sigma\sqrt{\sigma}}\right)\right] e^{\frac{-2}{\sqrt{\sigma}}(x-t)} = 3\left[-e^{\frac{-1}{\sqrt{\sigma}}(x-t)}\right]^2$

or $a_1\left(\frac{-6}{\sqrt{\sigma}}\right) = 3$ or $a_1 = \frac{-\sqrt{\sigma}}{2}$

Hence $U_1(x,t) = -\frac{\sqrt{\sigma}}{2} e^{\frac{-2}{\sqrt{\sigma}}(x-t)}$

Second-order term:

Let us choose $U_2 = a_2 e^{\frac{-3}{\sqrt{\sigma}}(x-t)}$

and substitute in

$$U_{2,t} + \sigma U_{2,xxx} = 6U_{0,x} U_{1,x}$$

Then $a_2\left[\frac{3}{\sqrt{\sigma}} + \sigma\left(\frac{-27}{\sigma\sqrt{\sigma}}\right)\right] e^{\frac{-3}{\sqrt{\sigma}}(x-t)} = 6e^{\frac{-1}{\sqrt{\sigma}}(x-t)}\left(-e^{\frac{-2}{\sqrt{\sigma}}(x-t)}\right)$

or $a_2 = \frac{\sqrt{\sigma}}{4}$

Hence $U_2(x,t) = \frac{\sqrt{\sigma}}{4} e^{\frac{-3}{\sqrt{\sigma}}(x-t)}$

By the method of mathematical induction, one can build in this way and obtain in general

$$U_n(x,t) = \left(-\frac{1}{2}\right)^n \sqrt{\sigma} e^{\frac{-(n+1)}{\sqrt{\sigma}}(x-t)}$$

Hence

$$U(x,t,\varepsilon) = \sqrt{\sigma} \sum_{n=0}^{\infty} e^{\frac{-1}{\sqrt{\sigma}}(x-t)} \left[-\frac{\varepsilon}{2} e^{\frac{-1}{\sqrt{\sigma}}(x-t)} \right]^n$$

$$= \frac{\sqrt{\sigma} e^{\frac{-1}{\sqrt{\sigma}}(x-t)}}{1 + \frac{\varepsilon}{2} e^{\frac{-1}{\sqrt{\sigma}}(x-t)}} \qquad (5)$$

and $V = \frac{\varepsilon \sqrt{\sigma} e^{\frac{-1}{\sqrt{\sigma}}(x-t)}}{1 + \frac{\varepsilon}{2} e^{\frac{-1}{\sqrt{\sigma}}(x-t)}}$

It is interesting to note that if

$$v = a + \left(\frac{b-a}{2\sqrt{\sigma}}\right) V = a + (b-a) \frac{\frac{\varepsilon}{2} e^{\frac{-1}{\sqrt{\sigma}}(x-t)}}{1 + \frac{\varepsilon}{2} e^{\frac{-1}{\sqrt{\sigma}}(x-t)}}$$

then v satisfies $v_t - \frac{6\sqrt{\sigma}}{b-a} v_x{}^2 + \sigma v_{xxx} = 0$

and exhibits shockwave in the limiting case

$$\lim_{\sigma \to 0} v(x,t) = \begin{Bmatrix} a & x-t > 0 \\ b & x-t < 0 \end{Bmatrix}$$

POTENTIAL KdV-BURGER EQUATION

Let us consider the potential KdV-Burger equation in the following form

$$u_t - \delta\, u_{xx} + \sigma u_{xxx} = 3u_x{}^2 \qquad (6)$$

The perturbed potential KdV-Burger equation is

$$U_t - \delta U_{xx} + \sigma U_{xxx} = 3\varepsilon U_x^{\ 2} \tag{7}$$

where $u = \varepsilon U$

By seeking a perturbation series solution

$$U(x,t,\varepsilon) = U_0(x,t) + \varepsilon U_1(x,t) + ... + \varepsilon^n U_n(x,t) + ...$$

one can break (7) into the following hierarchy of linear equations:

$$U_{0,t} - \delta U_{0,xx} + \sigma U_{0,xxx} = 0$$

$$U_{n,t} - \delta U_{n,xx} + \sigma U_{n,xxx} = 3\sum_{j=0}^{n-1} U_{n-1-j,x} U_{j,x}$$

n = 1, 2, 3, ………..

Zero-order term:

Let us choose the traveling wave

$$U_0 = \frac{1}{k} e^{-k(x-t)} \ ,$$

where $\sigma k^2 + \delta k - 1 = 0$. (8)

First-order term:

As before, let us choose

$$U_1 = a_1 e^{-2k(x-t)}$$

and substitute in

$$U_{1,t} - \delta U_{1,xx} + \sigma U_{1,xxx} = 3U_{0,x}^{\ 2}$$

Then using (8) we obtain

$$a_1 = \frac{-3}{2k(1+2\sigma k^2)} e^{-2k(x-t)}$$

And hence $U_1(x,t) = \dfrac{-3}{2k(1+2\sigma k^2)} e^{-2k(x-t)}$ (9)

$$U_{1,x}(x,t) = \frac{3}{(1+2\sigma k^2)} e^{-2k(x-t)}$$

Second-order term

Let us choose

$$U_2(x,t) = a_2 e^{-3k(x-t)}$$

and substitute in

$$U_{2,t} - \delta U_{2,xx} + \sigma U_{2,xxx} = 6U_{0,x}U_{1,x}$$

and using (8) we get

$$a_2 = \frac{3}{k(1+2\sigma k^2)(1+3\sigma k^2)}$$

$$= \left(-\frac{1}{2}\right)^2 \frac{3.4}{k(1+2\sigma k^2)(1+3\sigma k^2)}$$

$$U_2(x,t) = \left(-\frac{1}{2}\right)^2 \frac{3.4}{k(1+2\sigma k^2)(1+3\sigma k^2)} e^{-3k(x-t)} \tag{10}$$

Third-Order term:

Let us choose

$$U_3(x,t) = a_3 e^{-4k(x-t)}$$

and substituting in

$$U_{3,t} - \delta U_{3,xx} + \sigma U_{3,xxx} = 3(2U_{2,x}U_{0,x} + U_{1,x}{}^2$$

we get

$$U_3(x,t) = \left(-\frac{1}{2}\right)^3 \frac{3.4.5}{k(1+2\sigma k^2)(1+3\sigma k^2)(1+4\sigma k^2)} e^{-3k(x-t)} \times \frac{(27+63\sigma k^2)}{30(1+2\sigma k^2)} \tag{11}$$

Fourth-Order term:

Choosing $U_4(x,t) = a_4 e^{-5k(x-t)}$

and substituting in

$$U_{4,t} - \delta U_{4,xx} + \sigma U_{4,xxx} = 3(2U_{3,x}U_{0,x} + 2U_{2,x}U_{1,x})$$

we get

$$U_4(x,t) = \left(-\frac{1}{2}\right)^4 \frac{3.4.5.6}{k(1+2\sigma k^2)(1+3\sigma k^2)(1+4\sigma k^2)(1+5\sigma k^2)} e^{-5k(x-t)} \times \frac{(108+342\sigma k^2)}{150(1+2\sigma k^2)} \tag{12}$$

Similarly, the Fifth-order term gives

$$U_5(x,t) = \left(-\frac{1}{2}\right)^5 \frac{3.4.5.6.7}{k(1+2\sigma k^2)(1+3\sigma k^2)(1+4\sigma k^2)(1+5\sigma k^2)(1+6\sigma k^2)} e^{-6k(x-t)} \times$$

$$\frac{2}{1575} \frac{(54+171\sigma k^2)(1+2\sigma k^2)(1+3\sigma k^2) + 3(54+126\sigma k^2)(1+3\sigma k^2)(1+5\sigma k^2) + 81(1+2\sigma k^2)(1+4\sigma k^2)(1+5\sigma k^2)}{(1+2\sigma k^2)^2(1+3\sigma k^2)} \tag{13}$$

Numerical Calculation:

Choosing $\delta = 1$ and $\sigma = 1$ and using the relation (8) we obtain the value $k = \frac{-1+\sqrt{5}}{2}$ (golden ratio).

Now in (11), (12), (13) if we put the value of k in $\frac{(27+63\sigma k^2)}{30(1+2\sigma k^2)}$, $\frac{(108+342\sigma k^2)}{150(1+2\sigma k^2)}$,

$$\frac{2}{1575}\frac{(54+171\sigma k^2)(1+2\sigma k^2)(1+3\sigma k^2)+3(54+126\sigma k^2)(1+3\sigma k^2)(1+5\sigma k^2)+81(1+2\sigma k^2)(1+4\sigma k^2)(1+5\sigma k^2)}{(1+2\sigma k^2)^2(1+3\sigma k^2)}$$

we obtain 0.9649, 0.9019, 0.6496 respectively which can be approximated to 1.

With this we can conclude that

U_1, U_2, U_3, U_4, U_5 form a Hypergeometric solution of the equation (7) upto 5th term.

By the method of mathematical induction we can build in this way that

$$U_n(x,t) = \left(-\frac{1}{2}\right)^n \frac{3.4.......(n+2)}{k(1+2\sigma k^2)(1+3\sigma k^2).........(1+(n+1)\sigma k^2)} e^{-(n+1)k(x-t)}$$

Hence the perturbation series is

$$u(x,t,\varepsilon) = \frac{\varepsilon e^{-k(x-t)}}{k} \sum_{n=0}^{\infty} \frac{(3)_n (1)_n}{\left(\frac{1}{\sigma k^2}+2\right)_n (n)!} \left(\frac{-\varepsilon}{2\sigma k^2} e^{-k(x-t)}\right)^n$$

$$= \frac{\varepsilon e^{-k(x-t)}}{k} {}_2F_1\left(3,1;\frac{1}{\sigma k^2}+2;\frac{-\varepsilon}{2\sigma k^2} e^{-k(x-t)}\right)$$

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