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ON CONSTRUCTION OF HADAMARD MATRICES

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Abstract: A systematic study of Fourier matrix and construction of Hadamard matrices is presented. The paper presents a brief introduction to the Hadamard matrix and complex Hadamard matrices with its properties. Hadamard matrices find numerous applications in physics, engineering, coding theory, and in the field of quantum physics. The paper also presents an open problems for inequivalent complex Hadamard matrix of dimension seven.
Keywords: Hadamard Matrix, Orthogonal, Conjecture, Fourier Matrix.

1. Introduction

In 1867, James J. Sylvester introduced Hadamard matrices as a square matrix of order, $n \in \mathbb{R}$ with entries {-1, +1}, and later, it was further studied by Hadamard in 1893. It is defined as a square matrix H, of order, $n \in \mathbb{R}$ satisfying $HH^T = nI_n$, which has all the entries in the first row and first column $+1s$, and the rest of the elements are either $+1s$ or $-1s$ [1, 2]. And, a matrix which has all the entries in the first row and first column $+1s$, and the rest of the elements are either $+1s$ or $-1s$ or in terms of is are termed as complex Hadamard matrix. A Hadamard matrix is an orthogonal matrix satisfying the orthogonal property, i.e. the inner product of any two rows or columns is zero. A Hadamard matrices may be represented as [1-3],

$$H_1 = (1) \qquad H_2 = \begin{pmatrix} 1 & 1 \\ 1 & i^2 \end{pmatrix} \qquad H_4 = \begin{pmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{pmatrix} \qquad H_{2n} = \begin{pmatrix} H_n & H_n \\ H_n & -H_n \end{pmatrix}$$

and in general,

$$H_{2^k} = \begin{pmatrix} H_{2^{k-1}} & H_{2^{k-1}} \\ H_{2^{k-1}} & -H_{2^{k-1}} \end{pmatrix}$$
$$= H_2 \otimes H_{2^{k-1}}$$

where, $i^2 = -1, k \in \mathbb{N},\ I_n$ is Identity matrix of order n, and $\otimes$ is Kronecker Product.

Theorem 1 If $\boldsymbol{H}$ is a Hadamard matrix of order $\boldsymbol{n} \times \boldsymbol{n}$, then $\boldsymbol{HH^T} = \boldsymbol{n}\,\boldsymbol{I_n}$.

Proof The proof of this is simple and straightforward. ■

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Theorem 2 If $\boldsymbol{H}$ is a Hadamard matrix, and $\boldsymbol{H} = (\boldsymbol{h_{ij}})$ be a $\boldsymbol{n} \times \boldsymbol{n}$ matrix whose entries satisfy $|\boldsymbol{h_{ij}}| \leq \mathbf{1}$, then $|\boldsymbol{det}\ (\boldsymbol{H})| \leq \boldsymbol{n}^{\frac{n}{2}}$.

Proof Let $h_1,\ h_2,\ h_3\ \dots\ h_n$ be the rows of H. Using the simple Euclidean geometry, the volume of the parallelopiped with sides $h_1,\ h_2,\ h_3\ \dots\ h_n$ is given by $|det\ (H)|$, then

$$|\det (H)| \leq\ |\mathrm{h}_1|\ \dots |\mathrm{h}_\mathrm{n}| \tag{1}$$

where $|h_i|$ is the Euclidean length of h_i.

By hypothesis,

$$|h_i| = (h_{i1}^2 + \cdots + h_{in}^2)^{\frac{1}{2}}\ \leq\ n^{\frac{1}{2}}$$

if and only if $|h_{ij}| = 1$ for all j.

Therefore, for a Hadamard matrix of order n,

$$|\det (H)| \leq\ n^{\frac{n}{2}} \tag{2}$$

This completes the proof. ■

Theorem 3 If $\boldsymbol{H}$ is a Hadamard matrix of order $\boldsymbol{n}$, then $\boldsymbol{n}$ will be 1, 2 or multiple of 4, that is $\boldsymbol{n} = \mathbf{1}\ \boldsymbol{or}\ \mathbf{2}\ \boldsymbol{or}\ \boldsymbol{n} \equiv \mathbf{0}$ (mod 4).

Proof The proof of this is simple and straightforward [1, 2, 5]. ■

Theorem 3 is addressed as Hadamard conjecture. Despite the efforts of several mathematicians, this conjecture remains unproved even though it is believed to be true [3, 5].

2. Fourier Matrix

A Fourier matrix F is a square matrix of order $n \in \mathbb{R}$ is defined as

$$F_{jk} = \mathrm{e}^{2\pi\mathrm{i}\left(\frac{\mathrm{jk}}{\mathrm{n}}\right)} = \omega^{jk} \tag{3}$$

where $j, k = 0, 1, 2, 3 \dots n$, and $i = \sqrt{-1}$. When equation (3), multiplied by $\frac{1}{\sqrt{n}}$ the Fourier matrix is said to be normalized, and becomes unitary matrix [4].

Fourier matrix are written as,

$$\boldsymbol{F}_2 = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\mathbf{1} \end{pmatrix} \qquad \boldsymbol{F}_3 = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \boldsymbol{\omega} & \boldsymbol{\omega}^2 \\ \mathbf{1} & \boldsymbol{\omega}^2 & \boldsymbol{\omega} \end{pmatrix}; \boldsymbol{\omega} = \boldsymbol{e}^{\left(\frac{2\pi i}{3}\right)}$$

$$F_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{pmatrix} \qquad F_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 \\ 1 & \omega^2 & \omega^4 & \omega & \omega^3 \\ 1 & \omega^3 & \omega & \omega^4 & \omega^2 \\ 1 & \omega^4 & \omega^3 & \omega^2 & \omega \end{pmatrix}; \ \omega = e^{\left(\frac{2\pi i}{5}\right)}$$

In the similar manner, $F_6 \simeq F_2 \otimes F_3$, and $F_7$is a 7×7 matrix with $\omega = e^{\left(\frac{2\pi i}{7}\right)}$ and so on. But it is important to note that, $F_{24} \not\simeq F_4 \otimes F_6$ and $F_{64} \not\simeq F_8 \otimes F_8$ and so on [4, 6].

In general, a Fourier matrix may be represented as,

$$F_n = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & & \omega^{2(n-1)} \\ \vdots & & & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{pmatrix}; \quad \omega = e^{\left(\frac{2\pi i}{n}\right)} \tag{4}$$

3. Construction of Hadamard Matrices

Real Hadamard matrices obtained from a given matrix H by permuting or multiplying by -1 any of its rows or columns are said to be equivalent. For $n = 2, 4, 8\ and\ 12$ all real Hadamard matrices are equivalent.

A 2×2 Fourier matrix is equivalent to a 2×2 Hadamard matrix:

$$H_2 = F_2 = F_2^{(0)} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{5}$$

A 3×3 Fourier matrix is equivalent to a 3×3 Hadamard matrix:

$$H_3 = F_3 = F_3^{(0)} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}; \ \omega = e^{\left(\frac{2\pi i}{3}\right)} \tag{6}$$

A 4×4 Fourier matrix is equivalent to a 4×4 Hadamard matrix:

$$H_4 = F_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \tag{7}$$

Equation (7) may also be written in complex form as,

$$F_4^{(1)}(a)=\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i\exp(ia) & -1 & -i\exp(ia) \\ 1 & -1 & 1 & -1 \\ 1 & -i\exp(ia) & -1 & i\exp(ia) \end{pmatrix}; \text{ For } a=0,\ F_4^{(1)}(0)=F_4$$

A 5× 5 Fourier matrix is equivalent to a 5× 5 Hadamard matrix:

$$H_5=F_5=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 \\ 1 & \omega^2 & \omega^4 & \omega & \omega^3 \\ 1 & \omega^3 & \omega & \omega^4 & \omega^2 \\ 1 & \omega^4 & \omega^3 & \omega^2 & \omega \end{pmatrix};\ \omega=e^{\left(\frac{2\pi i}{5}\right)} \tag{8}$$

A 6× 6 Fourier matrix is equivalent to a 6× 6 Hadamard matrix:

$$H_6=F_6=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 \\ 1 & \omega^2 & \omega^4 & 1 & \omega^2 & \omega^4 \\ 1 & \omega^3 & 1 & \omega^3 & 1 & \omega^3 \\ 1 & \omega^4 & \omega^2 & 1 & \omega^4 & \omega^2 \\ 1 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \end{pmatrix};\ \omega=e^{\left(\frac{2\pi i}{6}\right)} \tag{9}$$

Equation (9) may also be written in equivalent form as symmetric matrix $\boldsymbol{D_6}$ given as,

$$D_6=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & i & -i & -i & i \\ 1 & i & -1 & i & -i & -i \\ 1 & -i & i & -1 & i & -i \\ 1 & -i & -i & i & -1 & i \\ 1 & i & -i & -i & i & -1 \end{pmatrix}$$

4. Applications

There are many applications of Hadamard matrices and complex Hadamard matrices. They are useful to construct bases of unitary operators, bases of entangled states and unitary depolarisers in quantum information theory [7]. These matrices allow to solve and construct error correcting codes [8], to find quantum designs and also to study spectral sets and Fuglede's conjecture [9].

5. Conclusions

In recent years, the knowledge of Hadamard matrices and complex Hadamard matrices has exponentially increased. Existence of Hadamard is ascertained by the existence of Fourier matrix of that dimension. However, an inequivalent complex Hadamard matrix of dimension five is known, but the complexity of the problem increases from dimension six [10]. This is due to the fact that six is not a prime power number, but there exists some other reasons too. It is an open problem to find inequivalent complex Hadamard matrices of dimension seven, where a one parametric family and a few number of single complex Hadamard matrices are known. Also, it is to know that whether a continuous family exists in dimension eleven or not. A complete understanding of Hadamard and complex Hadamard matrices could help in solving the Hadamard conjecture and the mutually unbiased bases problem in non-prime power dimensions.

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